IV. PROPAGATION EQUATIONS FOR INTEGRATED ENERGY-MOMENTUM AND SPIN AND ORBITAL ANGULAR MOMENTA

1. Introduction

Can one measure the non-metric parts of the connection which are present in the metric-connection theories of gravity? More specifically, might it be possible to experimentally measure the torsion field of a metric-Cartan connection theory? To answer these questions, one needs to investigate the behavior of test bodies and fields in the presence of background metric and connection fields.

The behavior of electromagnetic fields and other gauge fields was discussed in Section III.4 I showed that under the assumption of the standard minimal coupling procedure, these fields will ignore the torsion and behave as they do in a metric theory.

In Sections 2 and 3 of this chapter, I (in collaboration with William Stoeger) derive the propagation equations of the energy-momentum and angular momentum of a test body in the context of a metric-Cartan connection theory with dynamic torsion. The propagation equations show (Theorem IV.2 of Section 3 and Corollaries IV.1, IV.3 and IV.5 of Section 4) that the torsion couples to the elementary particle spin but not to the orbital angular momentum. Specifically, in the energy-momentum propagation equation (IV.3.16), the spin couples to the Cartan curvature while the orbital angular momentum couples to only the Christoffel curvature. Further, the covariant derivative of the spin couples to the defect tensor. In the angular momentum propagation equation (IV.3.17), the spin couples to the Cartan connection whereas the orbital angular momentum couples to only the Christoffel connection.

Consequently, if there is a torsion field, it cannot affect the precession of the Stanford gyroscope (Section 5a) but it can modify the precession of the angular momentum of a body with a net elementary particle spin. Unfortunately, for most bodies with both spin and orbital angular momentum, such as a magnetized iron gyroscope (Section 5b) or a neutron star (Section 5c), the spin is much smaller than the orbital angular momentum. Hence, the coupling of the spin to the Cartan connection and curvature is usually negligible compared to the coupling of the orbital angular momentum to the Christoffel connection and curvature.

Thus, the best way to search for torsion is to examine bodies with a net elementary particle spin but no orbital angular momentum. However even a non-rotating magnetized iron sphere (Section 5d) would have to have a radius of 10 m in order to have a spin angular momentum as large as the orbital angular momentum of the Stanford gyroscope. Putting such a large mass of iron into orbit is beyond present technology but may be possible in 50 years or so. Other systems with a net spin but no orbital angular momentum are mentioned in Section 5e.

Our method of deriving the propagation equations proceeds as follows: We first pick an arbitrary worldline, X(t), to represent the motion of the body and pick an arbitrary coordinate system x^a, centered on this worldline. In this coordinate system, we write out the canonical energy-momentum conservation law,

$$\nabla_b t^{ab} = \frac{1}{2} S^{dcb} \hat{R}_{cd}{}^{a}{}_{b} + t^{cd} \lambda_{cd}{}^{a}, \tag{1}$$

and the angular momentum conservation law,

$$\nabla_b S^{\hat{c}\hat{d}b} = 2\, t^{[cd]}. \tag{2}$$

Then we integrate various moments of these equations over the x^o-level surfaces. Finally, we solve the resulting equations for one propagation equation for the integrated canonical energy-momentum,

$$P^a = \int t^{ao} \sqrt{-\tilde{g}}\; d^3x, \tag{3}$$

and for a second propagation equation for both the integrated orbital angular momentum,

$$L^{ab} = \int (\delta x^a \, t^{bo} - \delta x^b \, t^{ao}) \sqrt{-\tilde{g}} \, d^3x, \tag{4}$$

and the integrated spin angular momentum,

$$S^{ab} = \int S^{abo} \sqrt{-\tilde{g}} \, d^3x \, . \tag{5}$$

This method of deriving propagation equations is similar to that used by Mathisson [1937], Papapetrou [1951] and Dixon [1970a,b, 1973, 1974] in the context of a metric theory of gravity. They derived propagation equations for the integrated metric energy-momentum,

$$\tilde{P}^a = \int \tilde{T}^{ab} \sqrt{-\tilde{g}} \, d^3x, \tag{6}$$

and the integrated metric angular-momentum,

$$\tilde{J}^{ab} = \int (\delta x^a \, \tilde{T}^{bo} - \delta x^b \, \tilde{T}^{ao}) \sqrt{-\tilde{g}} \, d^3x \, , \tag{7}$$

starting from the metric energy-momentum conservation law,

$$\nabla_b \tilde{T}^{ab} = 0. \tag{8}$$

In Corollaries IV.2, IV.4, IV.6 and IV.7 and the discussion in Section 4, I show that in the absence of torsion, our propagation equations correctly reduce to the metric theory results of Mathisson, Papapetrou and Dixon. These previous derivations of the metric theory results did not allow for the possibility that L^{ab} and S^{ab} might propagate differently, since they only considered the total $\tilde{J}^{ab}$ (which equals $L^{ab} + S^{ab}$ to lowest order in δx^a). However, Corollaries IV.4, IV.6 and IV.7 show that in a metric theory, L^{ab} and S^{ab} do in fact propagate identically.

In the presence of both torsion and spin, the conservation law (8) is no longer correct. Instead, one should use both of the conservation laws (1) and (2). Trautman [1972c] and Hehl [1971] have also derived propagation equations in the presence of torsion, but their method differs from ours. They also start from the conservation laws (1) and (2), but they do not define P^a, L^{ab} and S^{ab} as the integrals (3), (4), and (5). Instead they assume the energy-momentum tensor and spin tensor have the form,

$$t^{ab} = P^a v^b , \tag{9}$$

$$s^{abc} = S^{ab} v^c , \tag{10}$$

and substitute these equations into the conservation laws. I see no reason for assuming (9) and (10). In fact, this assumption implies that one is dealing with the restricted case when $L^{ab} = 0$. In that case, our propagation equations reduce (Corollary IV.5 of Section 4) to those found by Trautman and Hehl.

I wish to emphasize several points about the general propagation equations (IV.3.16) and (IV.3.17) of Theorem IV.2 in Section 3. First, the propagation equations are only appropriate to metric-Cartan connection theories with dynamics torsion. (The torsion is non-dynamic if it can be specified as a strictly local function of the spin tensor. For example, the ECSK theory has non-dynamic torsion.) The reason is that it would be inappropriate to treat a non-dynamic torsion field as a background field.

Second, we do not attempt either to find a preferred "center of mass" world line or to find equations of motion for this "center of mass." Rather, we find propagation equations for the energy-momentum and angular momentum along an arbitrary world-line (sufficiently close to the body) using an arbitrary coordinate system centered on that world-line. In the metric theory case, Madore [1966, 1969] and Beiglbock [1967] have constructed a "center of mass" world line and a generalized Fermi coordinate system based on that world line. In the metric-Cartan connection theory case, the Madore-Beiglbock construction still works but becomes ambiguous since one can use either Cartan or Christoffel geodesics and generalized Fermi transport rule in the construction.

Third, I want to emphasize that the propagation equations follow directly from the conservation laws (1) and (2). The conservation laws in turn may be either (i) derived via Noether's theorem from a scalar matter Lagrangian (Section III.5), or (ii) derived from a set of gravitational field equations with automatic conservation laws (Section V.3c), or (iii) assumed ad hoc. In any case, except for requiring the gravitational field equations to allow dynamic torsion and possibly to have automatic conservation laws, the propagation equations are completely independent of the choice of gravitational field equations.

Finally, before proceeding with the derivation of the propagation equations, I point out that the general formalism developed in Section 2 is appropriate to any type and number of charges describing the properties of the body. Thus it should be straightforward to generalize our results to find propation equations for (1) dilation current and hypermomentum in a metric-connection theory with a non-metric-compatible connection, (2) electric charge in a metric-connection theory with an electromagnetic field and (3) gauge charges in a metric-connection theory with a Yang-Mills field.

NOTE: Throughout the remainder of this chapter, my conventions on indices differ from the rest of the thesis. The conventions for this chapter are explained at the beginning of the next section.

2. General Framework

I first discuss notation and list my assumptions about the body and the nature of the surrounding spacetime, M. In this chapter, I need 3-dimensional indices (lower case Latin) as well as 4-dimentional indices (lower case Greek). I also use both orthonormal bases (indices with a carot) and coordinate bases (indices without a carot). In performing covariant derivatives, the orthonormal indices are corrected with the Cartan connection, while the coordinate indices are corrected with the Christoffel connection. (In the two tangent space formalism, one would regard the orthonormal bases as internal and the coordinate bases as external.) Upper case Latin indices denote any collection of indices, tangent or otherwise.

My first assumption is,

(i) The body is described by a collection of <u>current tensors</u>, $J_A^{\ \alpha}$. From these, the <u>current tensor densities</u> are defined as

$$\mathcal{J}_A^{\ \alpha} = \sqrt{-\tilde{g}}\ \ J_A^{\ \alpha}, \tag{1}$$

where $\tilde{g}$ is the determinant of the coordinate components of the metric, $g_{\alpha\beta}$. I assume,

(ii) The currents satisfy <u>differential "conservation" laws</u> which may be written as

$$\partial_\alpha \mathcal{J}_A^{\ \alpha} = \mathcal{F}_A\ , \tag{2}$$

where the $\mathcal{F}_A$ are <u>"sources"</u> which may or may not be tensorial and may or may not depend on the $\mathcal{J}_A^{\ \alpha}$.

The world tube of the body is defined to be the support of the currents; i.e.

$$\mathit{supp}\ \mathcal{J} = c\ell\{p\varepsilon M : \mathcal{J}_A^{\ \alpha}(p) \neq 0\}. \tag{3}$$

Here and hereafter, cl, int, and bd denoted the topological closure, interior and boundary of a subset of spacetime while ∂ denotes the boundary of a manifold with boundary. I assume

(iii) There exists a closed set, W, and a coordinate system, x^α, which satisfy

(a) the world tube of the body is contained in the interior of W, i.e.

$$\mathit{supp}\ \mathcal{J} \subset \mathit{int}\ W; \tag{4}$$

(b) the interior of W is dense in W, i.e.

$$W = \mathit{cl}\ \mathit{int}\ W; \tag{5}$$

(c) W is the union of timelike curves;

(d) the coordinate system, x^α, is defined on all of W;

(e) the coordinate basis, ∂_α, is oriented and time oriented, i.e. the basis vector, ∂_o, is everywhere timelike and future-directed, while the basis vectors, ∂_a, are everywhere space-like and oriented;

(f) the x^o-axis is entirely contained in int W;

(g) the x^o coordinate is an affine parameter on the x^o-axis;

(h) each level surface of x^o within W is compact.

Most of the following derivation works for any choice of W and x^α. A preferred choice may be specified later. For now I take W and x^α to be fixed but arbitrary satisfying assumption (iii). (Notice that if W and x^α were chosen to satisfy all of (iii) except (iii g) then it would be possible to rescale x^o to make all of (iii) satisfied.)

Assumptions (iii e, f, g) say that the x^o-axis is a future-directed, timelike curve, $X(t)$, which has affine parameter $t = x^o \in R$ and is contained in *int* W. The curve $X(t)$ has coordinates

$$X^o(t) = x^o(X(t)) = t, \; X^a(t) = x^a(X(t)) = 0,$$

or

$$X^\alpha(t) = t\,\delta^\alpha_o. \tag{6}$$

Since t is an affine parameter, the velocity vector,

$$v(t) = \frac{d}{dt} = \partial_o\Big|_{X(t)}, \tag{7}$$

is the unit tangent vector,

$$v^\alpha\, v_\alpha = s, \tag{8}$$

where the signature of the metric is (s, -s, -s, -s) and $s = \pm 1$. The coordinate components of the velocity vector are

$$v^\alpha(t) = \frac{d}{dt}\; X^\alpha(t) = \delta^\alpha_o, \tag{9}$$

so that equation (8) yields,

$$g_{oo}\Big|_{X(t)} = s\;. \tag{10}$$

Lowering the index, the coordinate components of the velocity 1-form are

$$v_\alpha(t) = g_{\alpha\beta}\, v^\beta = g_{\alpha o}\Big|_{X(t)}. \tag{11}$$

In particular, $v_o = s$. Notice that $v_a = g_{ao}\big|_{X(t)}$ is not necessarily zero because the x^o- axis is not assumed to be perpendicular to the x^o-level surfaces.

Assumptions (iii e, h) say that each x^o-level surface,

$$\Sigma(t) = \{p \in W : x^o(p) = t\} , \tag{12}$$

is spacelike and compact. Assumption (iii a) says

$$\mathcal{J}_A^{\ \alpha} = 0 \qquad \text{on} \quad \partial\Sigma(t) . \tag{13}$$

From the definitions (6) and (12), for each t,

$$X(t) \in \Sigma(t) . \tag{14}$$

In the following derivation I will often integrate various quantities over $\Sigma(t)$. So I will abbreviate:

$$\int f = \int_{\Sigma(t)} f(x)\, d^3x . \tag{15}$$

Such integrals will be regarded as functions of t defined along X(t). In particular, I will often need to compute the integral of a 4-dimensional divergence of the form $\int \partial_\alpha (f \mathcal{J}_A^{\ \alpha})$. By Stokes theorem and equation (13), this may be written as

$$\begin{aligned} \int \partial_\alpha (f\mathcal{J}_A^{\ \alpha}) &= \int \partial_o (f\mathcal{J}_A^{\ o}) + \int \partial_a (f\mathcal{J}_A^{\ a}) \\ &= \frac{d}{dt} \int f\mathcal{J}_A^{\ o} + \int_{\partial\Sigma} f\mathcal{J}_A^{\ a}\, d^2S_a \\ &= \frac{d}{dt} \int f\mathcal{J}_A^{\ o} . \end{aligned} \tag{16}$$

I will refer to equation (16) as "dropping the spatial divergences."

I also find it useful to introduce an orthonormal basis adapted to the foliation $\Sigma(t)$. Thus I assume

(iv) There is an oriented, time-oriented, orthonormal frame field, $e_{\hat{\mu}}$, defined on all of W such that the basis vector, $e_{\hat{o}}$, is everywhere orthogonal to $\Sigma(t)$ and future-directed and the basis vectors $e_{\hat{m}}$, are everywhere tangent to $\Sigma(t)$ and oriented.

Given W and x satisfying (iii) it is always possible to find a frame field, $e_{\hat{\mu}}$, satisfying (iv). Most of the following derivation works for any choice of $e_{\hat{\mu}}$. A preferred choice may be specified later. For now I take $e_{\hat{\mu}}$ to be fixed but arbitrary satisfying assumption (iv).

Since ∂_a and $e_{\hat{m}}$ are both bases for the tangent space to $\Sigma(t)$, one expands

$$e_{\hat{m}} = e_{\hat{m}}{}^a \, \partial_a . \tag{17}$$

If one then defines the lapse function, N, and the shift vector,

$$M = M^a \, \partial_a = M^{\hat{m}} \, e_{\hat{m}}, \tag{18}$$

by requiring

$$\partial_o = N \, e_{\hat{o}} + M , \tag{19}$$

one finds that the bases are related by

$$\begin{aligned} e_{\hat{o}} &= \frac{1}{N} \partial_o - \frac{M^a}{N} \partial_a, & \theta^{\hat{o}} &= N \, dx^o, \\ e_{\hat{m}} &= e_{\hat{m}}{}^a \, \partial_a, & \theta^{\hat{m}} &= M^{\hat{m}} \, dx^o + \theta^{\hat{m}}{}_a \, dx^a, \end{aligned} \tag{20}$$

and conversely,

$$\partial_o = N\, e_{\hat{o}} + M^{\hat{m}}\, e_{\hat{m}}, \qquad dx^o = \frac{1}{N}\, \theta^{\hat{o}},$$

$$\partial_a = \theta^{\hat{m}}{}_a\, e_{\hat{m}}, \qquad dx^a = -\frac{M^a}{N}\, \theta^{\hat{o}} + e_{\hat{m}}{}^a\, \theta^{\hat{m}}, \tag{21}$$

where $\theta^{\hat{m}}{}_a$ is the inverse matrix of $e_{\hat{m}}{}^a$:

$$\theta^{\hat{m}}{}_a\, e_{\hat{m}}{}^b = \delta_a{}^b, \qquad \theta^{\hat{m}}{}_a\, e_{\hat{n}}{}^a = \delta^{\hat{m}}{}_{\hat{n}}. \tag{22}$$

From (17), (18) and (22), one finds

$$M^a = M^{\hat{m}}\, e_{\hat{m}}{}^a, \qquad M^{\hat{m}} = M^a\, \theta^{\hat{m}}{}_a\, . \tag{23}$$

Notice that the 3-dimensional 1-form basis dual to $e_{\hat{m}}$ is

$${}^3\theta^{\hat{m}} = \theta^{\hat{m}}{}_a\, dx^a\, , \tag{24}$$

rather than $\theta^{\hat{m}}$ from equation (20).

From the formulas

$$g_{\alpha\beta} = \eta_{\hat{\mu}\hat{\nu}}\, \theta^{\hat{\mu}}{}_\alpha\, \theta^{\hat{\nu}}{}_\beta\, , \qquad g^{\alpha\beta} = \eta^{\hat{\mu}\hat{\nu}}\, e_{\hat{\mu}}{}^\alpha\, e_{\hat{\nu}}{}^\beta\, , \tag{25}$$

where

$$\eta = \mathrm{diag}(s,-s,-s,-s), \qquad s = \pm 1, \tag{26}$$

one finds that the coordinate components of the metric are

$$g_{oo} = s\, N^2 - s\, \delta_{\hat{m}\hat{n}}\, M^{\hat{m}}\, M^{\hat{n}},$$

$$g_{ab} = -\, s\, \delta_{\hat{m}\hat{n}}\, \theta^{\hat{m}}{}_a\, \theta^{\hat{n}}{}_b\, , \tag{27}$$

$$g_{ao} = -\, s\, \delta_{\hat{m}\hat{n}}\, \theta^{\hat{m}}{}_a\, M^{\hat{n}} = g_{ab}\, M^b,$$

and the coordinate components of the inverse metric are

$$g^{oo} = s\,\frac{1}{N^2}\ ,$$

$$g^{ab} = s\ \frac{M^a M^b}{N^2} - s\ \delta^{\hat{m}\hat{n}}\ e_{\hat{m}}{}^a\ e_{\hat{n}}{}^b\ , \tag{28}$$

$$g^{ao} = -\ s\ \frac{M^a}{N^2}\ ,$$

It is conventional to introduce the orthonormal projection operator

$$\gamma = -\ s\ g + \theta^{\hat{o}} \otimes \theta^{\hat{o}} = \delta_{\hat{m}\hat{n}}\ \theta^{\hat{m}} \otimes \theta^{\hat{n}}, \tag{29}$$

which has coordinate components

$$\gamma_{oo} = \delta_{\hat{m}\hat{n}}\ M^{\hat{m}}\ M^{\hat{n}}\ ,$$

$$\gamma_{ab} = \delta_{\hat{m}\hat{n}}\ \theta^{\hat{m}}{}_a\ \theta^{\hat{n}}{}_b\ , \tag{30}$$

$$\gamma_{ao} = \delta_{\hat{m}\hat{n}}\ \theta^{\hat{m}}{}_a\ M^{\hat{n}}\ .$$

Notice that γ differs from the 3-dimensional metric,

$$^3\gamma = \delta_{\hat{m}\hat{n}}\ {}^3\theta^{\hat{m}} \otimes {}^3\theta^{\hat{n}} = \gamma_{ab}\ dx^a \otimes dx^b$$

$$= \delta_{\hat{m}\hat{n}}\ \theta^{\hat{m}}{}_a\ \theta^{\hat{n}}{}_b\ dx^a \otimes dx^b, \tag{31}$$

which has a 3-dimensional inverse,

$$\gamma^{-1} = \delta^{\hat{m}\hat{n}}\ e_{\hat{m}} \otimes e_{\hat{n}} = \gamma^{ab}\ \partial_a \otimes \partial_b$$

$$= \delta^{\hat{m}\hat{n}}\ e_{\hat{m}}{}^a\ e_{\hat{n}}{}^b\ \partial_a \otimes \partial_b\ . \tag{32}$$

For the computations in this chapter, I find it also useful to introduce a coordinate projection operator

$$\rho = \mathbb{1} - \partial_o \otimes dx^o = \partial_a \otimes dx^a , \tag{33}$$

which has coordinate components

$$\rho^\alpha{}_\beta = \delta^\alpha_\beta - \delta^\alpha_o \, \delta^o_\beta = \delta^\alpha_a \, \delta^a_\beta . \tag{34}$$

Along the curve, X(t), equations (7) and (9) say that the velocity vector, v, coincides with the coordinate basis vector, ∂_o. I also introduce the normal vector,

$$u(t) = e_{\hat{o}}\big|_{X(t)} , \tag{35}$$

with coordinate components, from (20)

$$u^\alpha(t) = \frac{1}{N} \delta^\alpha_o - \frac{M^a}{N} \delta^\alpha_a . \tag{36}$$

Lowering the index yields the coordinate components of the normal 1-form,

$$u_\alpha(t) = s \; N \; \delta^o_\alpha . \tag{37}$$

Thus as a 1-form

$$u(t) = s \; N \; dx^o\big|_{X(t)} = s \; \theta^{\hat{o}}\big|_{X(t)} . \tag{38}$$

The velocity, v, and normal vector, u, are related by

$$v(t) = N(t) \; u(t) + M(t). \tag{39}$$

Along X(t), the coordinate projection operator is

$$\rho = \mathbb{1} - s \, \frac{1}{N} v \otimes u , \tag{40}$$

with coordinate components

$$\rho^{\alpha}{}_{\beta} = \delta^{\alpha}{}_{\beta} - s\,\frac{1}{N}\,v^{\alpha}\,u_{\beta} = \delta^{\alpha}{}_{\beta} - v^{\alpha}\,\delta^{o}{}_{\beta}\,. \tag{41}$$

Further, the zero components of tensors may be written as either

$$T^{o} = s\,\frac{1}{N}\,T^{\alpha}\,u_{\alpha}\,, \tag{42}$$

or

$$T_{o} = T_{\alpha}\,v^{\alpha}\,. \tag{43}$$

In the following derivation I will need to power series expand various quantities within $\Sigma(t)$ about $X(t)$. Thus it is useful to introduce a new coordinate system, δx^{α}, on each surface, $\Sigma(t)$, centered at $X(t)$. At $p \in \Sigma(t)$,

$$\delta x^{\alpha}(p) = x^{\alpha}(p) - X^{\alpha}(t), \tag{44}$$

or

$$\delta x^{o}(p) = 0, \quad \delta x^{a}(p) = x^{a}(p). \tag{45}$$

My final assumption is

(v) On each surface, $\Sigma(t)$, the components of the orthonormal frames, $\theta^{\hat{\mu}}{}_{\alpha}$, and the defect, $\lambda^{\alpha}{}_{\beta\gamma}$, may be expanded about $X(t)$ in a power series in δx^{α}:

$$x^{\alpha} = X^{\alpha} + \delta x^{\alpha}, \tag{46}$$

$$\theta^{\hat{\mu}}{}_{\alpha}\big|_{x} = \theta^{\hat{\mu}}{}_{\alpha}\big|_{X} + \delta x^{\beta}\,\partial_{\beta}\,\theta^{\hat{\mu}}{}_{\alpha}\big|_{X} + \dots\,, \tag{47}$$

$$\lambda^{\alpha}{}_{\beta\gamma}\big|_{x} = \lambda^{\alpha}{}_{\beta\gamma}\big|_{X} + \delta x^{\varepsilon}\,\partial_{\varepsilon}\,\lambda^{\alpha}{}_{\beta\gamma}\big|_{X} + \dots\,. \tag{48}$$

Hence, the coordinate components of the metric, $g_{\alpha\beta}$, the Christoffel connection, $\{^{\alpha}_{\beta\gamma}\}$, the Christoffel curvature, $\tilde{R}^{\alpha}{}_{\beta\gamma\delta}$, the torsion, $Q^{\alpha}{}_{\beta\gamma}$, the Cartan connection, $\Gamma^{\alpha}{}_{\beta\gamma}$, and the Cartan curvature, $\hat{R}^{\alpha}{}_{\beta\gamma\delta}$, may also be expanded.

I can now introduce the n-th integrated moments of the currents which are defined as

$$K_A{}^{\alpha_1 \cdots \alpha_n \beta} = \int \delta x^{\alpha_1} \cdots \delta x^{\alpha_n} \mathcal{J}_A{}^{\beta}$$

$$= \int \left(\prod_{j=1}^{n} \delta x^{\alpha_j} \right) \mathcal{J}_A{}^{\beta}, \tag{49}$$

for $n \geq 0$. These are symmetric in the indices $\alpha_1 \ldots \alpha_n$, and zero if any of the indices $\alpha_1 \ldots \alpha_n$ are zero. Certain moments have additional names. The integrated charges are

$$Q_A = K_A{}^{o} = \int \mathcal{J}_A{}^{o}, \tag{50}$$

and the n-th integrated moments of the charge are

$$Q_A{}^{\alpha_1 \cdots \alpha_n} = K_A{}^{\alpha_1 \cdots \alpha_n o} = \int \left(\prod_{j=1}^{n} \delta x^{\alpha_j} \right) \mathcal{J}_A{}^{o}. \tag{51}$$

The integrated moments of the currents and charges are to be regarded as tensors which depend on the choice of coordinates, x^{α}, are defined only along the curve X, and are specified in the coordinate basis, ∂_{α}. The goal of this chapter is to find equations of motion for the integrated charge moments which describe how these quantities change along the curve X(t). The technique is to integrate over $\Sigma(t)$ all possible moments of the differential conservation laws (2) and to attempt to solve the resulting equations for $\frac{d}{dt} Q_A{}^{\alpha_1 \cdots \alpha_n}$ while eliminating the other (spatial) components of $K_A{}^{\alpha_1 \cdots \alpha_n \beta}$. The crucial equation in this procedure follows from the identity proved in the following theorem.

Theorem IV.1:

For any integer $n \geq 0$,

$$\frac{d}{dt}\int\Big(\prod_{j=1}^{n}\delta x^{\alpha_j}\Big)\mathcal{J}_A^{\,o}+\sum_{i=1}^{n}v^{\alpha_i}\int\Big(\prod_{\substack{j=1\\ \neq i}}^{n}\delta x^{\alpha_j}\Big)\mathcal{J}_A^{\,o}$$

$$=\sum_{i=1}^{n}\int\Big(\prod_{\substack{j=1\\ \neq i}}^{n}\delta x^{\alpha_j}\Big)\mathcal{J}_A^{\,\alpha_i}+\int\Big(\prod_{j=1}^{n}\delta x^{\alpha_j}\Big)\partial_\beta\mathcal{J}_A^{\,\beta}. \qquad (52)$$

Proof:

The proof proceeds by induction on n. For $n = 0$, (52) becomes

$$\frac{d}{dt}\int\mathcal{J}_A^{\,o}=\int\partial_\beta\mathcal{J}_A^{\,\beta}, \qquad (53)$$

which follows by dropping the spatial divergence.

For $n \geq 1$, let $N = \{1, \ldots, n\}$ and if $i \in N$, let $N_i = N - \{i\}$. Similarly, if S is any subset of N and if $i \in S$, let $S_i = S - \{i\}$. Assume that if S is a *proper* subset of N, then

$$\frac{d}{dt}\int\Big(\prod_{j\in S}\delta x^{\alpha_j}\Big)\mathcal{J}_A^{\,o}+\sum_{i\in S}v^{\alpha_i}\int\Big(\prod_{j\in S_i}\delta x^{\alpha_j}\Big)\mathcal{J}_A^{\,o}$$

$$=\sum_{i\in S}\int\Big(\prod_{j\in S_i}\delta x^{\alpha_j}\Big)\mathcal{J}_A^{\,\mu_i}+\int\Big(\prod_{j\in S}\delta x^{\alpha_j}\Big)\partial_\beta\mathcal{J}_A^{\,\beta}. \qquad (54)$$

Multiply (54) by $\prod_{j\in N-S} X^{\alpha_j}$ and sum over all proper subsets S of N:

(Note: if $i\in S$ then $N-S=N_i-S_i$.)

$$\sum_{S\subsetneq N} \Big(\prod_{j\in N-S} X^{\alpha_j}\Big) \frac{d}{dt}\int \Big(\prod_{j\in S}\delta x^{\alpha_j}\Big) \mathscr{g}_A^{\,o}$$

$$+ \sum_{S\subsetneq N}\ \sum_{i\in S} v^{\alpha_i}\Big(\prod_{j\in N_i-S_i} X^{\alpha_j}\Big)\int\Big(\prod_{j\in S_i}\delta x^{\alpha_j}\Big)\mathscr{g}_A^{\,o}$$

$$= \sum_{S\subsetneq N}\ \sum_{i\in S} \Big(\prod_{j\in N_i - S_i} X^{\alpha_j}\Big)\int\Big(\prod_{j\in S_i}\delta x^{\alpha_j}\Big)\mathscr{g}_A^{\,\alpha_i}$$

$$+\sum_{S\subsetneq N}\Big(\prod_{j\in N-S}X^{\alpha_j}\Big)\int\Big(\prod_{j\in S}\delta x^{\alpha_j}\Big)\partial_\beta \mathscr{g}_A^{\,\beta} . \tag{55}$$

On the other hand, there is the identity

$$\partial_\beta\Big[\Big(\prod_{j\in N} x^{\alpha_j}\Big)\mathscr{g}_A^{\,\beta}\Big] = \sum_{i\in N}\Big(\prod_{j\in N_i} x^{\alpha_j}\Big)\mathscr{g}_A^{\,\alpha_i} + \Big(\prod_{j\in N} x^{\alpha_j}\Big)\partial_\beta \mathscr{g}_A^{\,\beta} . \tag{56}$$

Integrate (56) over $\Sigma(t)$ and drop a spatial divergence:

$$\frac{d}{dt}\int\Big(\prod_{j\in N} x^{\alpha_j}\Big)\mathscr{g}_A^{\,o} = \sum_{i\in N}\int\Big(\prod_{j\in N_i} x^{\alpha_j}\Big)\mathscr{g}_A^{\,\alpha_i} + \int\Big(\prod_{j\in N} x^{\alpha_j}\Big)\partial_\beta \mathscr{g}_A^{\,\beta} . \tag{57}$$

Substitute $x^{\alpha_j} = X^{\alpha_j} + \delta x^{\alpha_j}$, using

$$\prod_{j\in N} x^{\alpha_j} = \sum_{S\subset N}\Big(\prod_{j\in N-S} X^{\alpha_j}\Big)\Big(\prod_{j\in S}\delta x^{\alpha_j}\Big) , \tag{58}$$

$$\prod_{j\in N_i} x^{\alpha_j} = \sum_{T\subset N_i}\Big(\prod_{j\in N_i-T} X^{\alpha_j}\Big)\Big(\prod_{j\in T}\delta x^{\alpha_j}\Big) , \tag{59}$$

and the fact that X^{α_j} is constant on the surface $\Sigma(t)$. The result is:

$$\sum_{S\subset N} \frac{d}{dt} [(\prod_{j\varepsilon N-S} X^{\alpha_j}) \int (\prod_{j\varepsilon S} \delta x^{\alpha_j}) \mathcal{J}_A{}^{o}]$$

$$= \sum_{i\varepsilon N} \sum_{T\subset N_i} (\prod_{j\varepsilon N_i - T} X^{\alpha_j}) \int (\prod_{j\varepsilon T} \delta x^{\alpha_j}) \mathcal{J}_A{}^{\alpha_i}$$

$$+ \sum_{S\subset N} (\prod_{j\varepsilon N-S} X^{\alpha_j}) \int (\prod_{j\varepsilon S} \delta x^{\alpha_j}) \partial_\beta \mathcal{J}_A{}^{\beta} . \qquad (60)$$

Note, the summation $\sum_{i\varepsilon N} \sum_{T\subset N_i}$ may be replaced by $\sum_{S\subset N} \sum_{i\varepsilon S}$ where $S = T \cup \{i\}$ and $T = S_i$. Make this replacement in (60). Also, on the left hand side of (60), perform the time derivative using (9) and replace the dummy index S by T in one term:

$$\sum_{S\subset N} (\prod_{j\varepsilon N-S} X^{\alpha_j}) \frac{d}{dt} \int (\prod_{j\varepsilon S} \delta x^{\alpha_j}) \mathcal{J}_A{}^{o}$$

$$+ \sum_{T\subset N} \sum_{i\varepsilon N-T} v^{\alpha_i} (\prod_{j\varepsilon N_i - T} X^{\alpha_j}) \int (\prod_{j\varepsilon T} \delta x^{\alpha_j}) \mathcal{J}_A{}^{o}$$

$$= \sum_{S\subset N} \sum_{i\varepsilon S} (\prod_{j\varepsilon N_i - S_i} X^{\alpha_j}) \int (\prod_{j\varepsilon S_i} \delta x^{\alpha_j}) \mathcal{J}_A{}^{\alpha_i}$$

$$+ \sum_{S\subset N} (\prod_{j\varepsilon N-S} X^{\alpha_j}) \int (\prod_{j\varepsilon S} \delta x^{\alpha_j}) \partial_\beta \mathcal{J}_A{}^{\beta} . \qquad (61)$$

Note, the summation $\sum_{T\subset N} \sum_{i\varepsilon N-T}$ may be replaced by $\sum_{S\subset N} \sum_{i\varepsilon S}$ where $S = T \cup \{i\}$ and $T = S_i$. Thus the second term on the left becomes

$$\sum_{S\subset N} \sum_{i\varepsilon S} v^{\alpha_i} (\prod_{j\varepsilon N_i - S_i} X^{\alpha_j}) \int (\prod_{j\varepsilon S_i} \delta x^{\alpha_j}) \mathcal{J}_A{}^{o} . \qquad (62)$$

Finally, subtract (55) from (61) using (62). The only terms which do not cancel are those for which S = N. These terms are just (52).

Q. E. D.

Using the definitions (49) and (51) of the integrated moments, the definition (34) of the projection operator, and conservation laws (2), the identity (52) becomes

$$\frac{d}{dt} Q_A^{\alpha_1 \ldots \alpha_n} = \sum_{i=1}^{n} K_A^{\alpha_1 \ldots \not{\alpha}_i \ldots \alpha_n \beta} \rho^{\alpha_i}{}_{\beta} + \int \delta x^{\alpha_1} \ldots \delta x^{\alpha_n} \mathcal{J}_A , \tag{63}$$

for any integer $n \geq 0$. Here, the slash, /, through the index, α_i, indicates that this index is deleted from the list. For n = 0, 1, 2 equation (63) reduces to

$$\frac{d}{dt} Q_A = \int \mathcal{J}_A , \tag{64}$$

$$\frac{d}{dt} Q_A^{\alpha} = K_A^{\beta} \rho^{\alpha}{}_{\beta} + \int \delta x^{\alpha} \mathcal{J}_A , \tag{65}$$

$$\frac{d}{dt} Q_A^{\alpha\beta} = K_A^{\beta\gamma} \rho^{\alpha}{}_{\gamma} + K_A^{\alpha\gamma} \rho^{\beta}{}_{\gamma} + \int \delta x^{\alpha} \delta x^{\beta} \mathcal{J}_A . \tag{66}$$

By examining the proof of Theorem IV.1, one can convince oneself that equations (63) contain all of the information about the integrated moments that can be obtained from the conservations laws.

For a given theory with a specific set of conservation laws, the procedure is (a) to assume that only a finite number of the integrated moments are non-negligible, (b) to expand the sources, $\mathcal{J}_A$, in a power series about X(t) and to substitute these expansions into equations (63), and (c) to attempt to solve the resulting equations for $\frac{d}{dt} Q_A^{\alpha_1 \ldots \alpha_n}$ and $K_A^{\alpha_1 \ldots \alpha_n \beta} \rho^{\alpha}{}_{\beta}$ in terms of $Q_A^{\alpha_1 \ldots \alpha_n}$ and the background geometry. This would produce a consistent set of equations of motion for evolving the non-negligible charge and current moments from initial data.

In the next section, I carry out this procedure for the energy-momentum and angular-momentum conservation laws in a metric-Cartan connection theory of gravity. In principle the same procedure should work for the conservation of dilation current and hypermomentum when the connection is not assumed to be metric-compatible, and should also work for the conservation of electric gauge charges in any gauge theory. I have not carried out these computations.

3. Propagation Equations in a Metric-Cartan Connection Theory

In Section III.5.c I derived conservation laws in the context of metric-Cartan connection theories of gravity with an $O_\circ(3,1,R)$-internal tangent bundle. The energy-momentum conservation law is equation (III.5.81),

$$\nabla_\delta t_{\hat{\mu}}{}^{\delta} = Q^{\alpha}{}_{\hat{\mu}\delta} t_\alpha{}^{\delta} + \frac{1}{2} \hat{R}^{\alpha}{}_{\beta\hat{\mu}\delta} S^{\beta}{}_{\alpha}{}^{\delta} \quad , \tag{1}$$

and the angular-momentum conservation law is equation (III.5.72),

$$\nabla_\alpha S^{\hat{\mu}}{}_{\hat{\nu}}{}^{\alpha} = t^{\hat{\mu}}{}_{\hat{\nu}} - t_{\hat{\nu}}{}^{\hat{\mu}} \quad . \tag{2}$$

By introducing the tensor densities

$$\mathcal{t}^{\gamma\delta} = \sqrt{-\tilde{g}} \;\; t^{\gamma\delta} \; , \tag{3}$$

$$\mathcal{S}^{\gamma\delta\alpha} = \sqrt{-\tilde{g}} \; S^{\gamma\delta\alpha}, \tag{4}$$

the conservation laws become

$$\partial_\delta \mathcal{t}^{\gamma\delta} = (\lambda_{\alpha\delta}{}^{\gamma} - \{^{\gamma}{}_{\alpha\delta}\})\mathcal{t}^{\alpha\delta} + \frac{1}{2} \hat{R}_{\alpha\beta}{}^{\gamma}{}_{\delta} \mathcal{S}^{\beta\alpha\delta} \; , \tag{5}$$

$$\partial_\alpha \mathcal{S}^{\gamma\delta\alpha} = 2\mathcal{t}^{[\gamma\delta]} + 2\Gamma^{[\gamma}{}_{\beta\alpha} \mathcal{S}^{\delta]\beta\alpha} \; . \tag{6}$$

The n-th integrated moments of $\mathcal{t}^{\gamma\delta}$ and $\mathcal{S}^{\gamma\delta\alpha}$ are

$$M^{\beta_1 \ldots \beta_n \gamma\delta} = \int \delta x^{\beta_1} \ldots \delta x^{\beta_n} \, \mathcal{t}^{\gamma\delta} \quad , \tag{7}$$

$$N^{\beta_1 \ldots \beta_n \gamma\delta\alpha} = \int \delta x^{\beta_1} \ldots \delta x^{\beta_n} \, \mathcal{S}^{\gamma\delta\alpha} \quad . \tag{8}$$

These are symmetric in $\beta_1 \ldots \beta_n$ and zero if any of $\beta_1 \ldots \beta_n$ are zero. Further, $N^{\beta_1 \ldots \beta_n \gamma\delta\alpha}$ is antisymmetric in γ and δ. Certain moments have additional names. The integrated energy-momentum is

$$P^{\gamma} = M^{\gamma o} = \int t^{\gamma o}. \tag{9}$$

The integrated orbital angular-momentum is

$$L^{\gamma\delta} = M^{\gamma\delta o} - M^{\delta\gamma o} = \int (\delta x^{\gamma}\, t^{\delta o} - \delta x^{\delta}\, t^{\gamma o}) . \tag{10}$$

The integrated spin angular-momentum is

$$S^{\gamma\delta} = N^{\gamma\delta o} = \int \mathcal{S}^{\gamma\delta o}. \tag{11}$$

The integrated moments are to be regarded as tensors defined along the curve, X(t), and specified in the coordinate basis, ∂_{α}. As such, their indices are lowered using the coordinate components of the metric at X(t), e.g.

$$P_{\delta} = g_{\delta\gamma}\big|_{X(t)}\, P^{\gamma}. \tag{12}$$

Since $g_{\delta\gamma}$ is not constant on $\Sigma(t)$, this yields a different result than computing

$$\int t_{\delta}{}^{o} = \int g_{\delta\gamma}\, t^{\gamma o} , \tag{13}$$

which could alternatively be used as the definition of the integrated energy-momentum instead of (9). This ambiguity in the definition of the integrated moments was pointed out by Madore and leads to ambiguities (discussed below) in the interpretation of the evolution equations derived below. I feel these ambiguities could be removed by using $t_{\hat{\mu}}{}^{\alpha}$ and $\mathcal{S}^{\hat{\mu}}{}_{\hat{\nu}}{}^{\alpha}$ ($\hat{\mu}$ and $\hat{\nu}$ orthonormal, α coordinate) in definitions (7) through (11). However, I have not yet redone the calculation of the evolution equations. So the following calculation is done using the *contravariant coordinate* components of $t^{\gamma\delta}$ and $\mathcal{S}^{\gamma\delta\alpha}$ in definitions (7) through (11).

I now apply the derivation in the previous section to the conservation laws (5) and (6). For the integrated moments of $t^{\gamma\delta}$ and $S^{\gamma\delta\alpha}$, equation (IV.2.63) becomes

$$\frac{d}{dt} M^{\beta_1 \cdots \beta_n \gamma o} = \sum_{i=1}^{n} (M^{\beta_1 \cdots \not\beta_i \cdots \beta_n \gamma \beta_i} - v^{\beta_i} M^{\beta_1 \cdots \not\beta_i \cdots \beta_n \gamma o})$$

$$+ \int \delta x^{\beta_1} \ldots \delta x^{\beta_n} [(\lambda_{\alpha\delta}{}^{\gamma} - \{^{\gamma}{}_{\alpha\delta}\}) t^{\alpha\delta} + \frac{1}{2} \hat{R}_{\alpha\beta}{}^{\gamma}{}_{\delta} S^{\beta\alpha\delta}], \qquad (14)$$

$$\frac{d}{dt} N^{\beta_1 \cdots \beta_n \gamma\delta o} = \sum_{i=1}^{n} (N^{\beta_1 \cdots \not\beta_i \cdots \beta_n \gamma\delta\beta_i} - v^{\beta_i} N^{\beta_1 \cdots \not\beta_i \cdots \beta_n \gamma\delta o})$$

$$+ \int \delta x^{\beta_1} \ldots \delta x^{\beta_n} (2\, t^{[\gamma\delta]} + 2\, \Gamma^{[\gamma}{}_{\beta\alpha} S^{\delta]\beta\alpha}). \qquad (15)$$

I will use equations (14) and (15) to find equations of motion of bodies with and without orbital angular momentum and with and without spin angular momentum, in the context of both metric theories of gravity and metric-Cartan connection theories of gravity. The most general of the eight cases is presented in the following theorem. The other cases appear as corollaries in the next section.

Theorem IV.2:

In a metric-Cartan connection theory of gravity, if the integrated moments, $M^{\beta_1 \dots \beta_n \gamma\delta}$ *with* $n \geq 2$ *and* $N^{\beta_1 \dots \beta_n \gamma\delta\alpha}$ *with* $n \geq 1$, *are negligible compared to* $M^{\gamma\delta}$, $M^{\beta\gamma\delta}$ *and* $N^{\gamma\delta\alpha}$, *then*

$$\nabla_v p^\gamma = \frac{1}{2} L^{\beta\alpha} \tilde{R}_{\alpha\beta}{}^{\gamma}{}_{\delta} v^\delta + \frac{1}{2} S^{\beta\alpha} \hat{R}_{\alpha\beta}{}^{\gamma}{}_{\delta} v^\delta + \frac{1}{2} \lambda_{\alpha\beta}{}^{\gamma} e_{\hat{\kappa}}{}^{\alpha} e_{\hat{\lambda}}{}^{\beta} \nabla_v S^{\hat{\kappa}\hat{\lambda}} + \frac{1}{2} \rho^{\delta}{}_{\nu} N^{\beta\alpha\nu} g^{\gamma\varepsilon} \nabla_\varepsilon \lambda_{\alpha\beta\delta} , \tag{16}$$

$$\nabla_v L^{\gamma\delta} + e_{\hat{\kappa}}{}^{\gamma} e_{\hat{\lambda}}{}^{\delta} \nabla_v S^{\hat{\kappa}\hat{\lambda}} = -2 v^{[\gamma} p^{\delta]} + \lambda_{\alpha\beta}{}^{[\gamma} \rho^{\delta]}{}_{\nu} N^{\alpha\beta\nu} + 2 \lambda^{[\gamma}{}_{\beta\alpha} N^{\delta]\beta\nu} \rho^{\alpha}{}_{\nu} , \tag{17}$$

$$M^{\gamma\delta} = v^\delta P^\gamma + \frac{d}{dt} \left(\frac{1}{2} L^{\delta\gamma} + v^{(\delta} L^{\gamma)o} - \rho^{(\delta}{}_{\nu} N^{\gamma)o\nu}\right) + \frac{1}{2} \lambda_{\alpha\beta}{}^{\gamma} \rho^{\delta}{}_{\nu} N^{\alpha\beta\nu} - \{{}^{\gamma}{}_{\alpha\beta}\} \rho^{\delta}{}_{\mu} \left(v^\alpha L^{\beta\mu} - \rho^{\alpha}{}_{\nu} N^{\beta\mu\nu}\right), \tag{18}$$

$$M^{\beta\gamma\delta} = -\rho^{\beta}{}_{\mu} \left(\frac{1}{2} N^{\gamma\delta\mu} + v^{(\gamma} L^{\delta)\mu} - \rho^{(\gamma}{}_{\nu} N^{\delta)\mu\nu}\right), \tag{19}$$

$$N^{\gamma\delta\alpha} = v^\alpha S^{\gamma\delta} + \rho^{\alpha}{}_{\nu} N^{\gamma\delta\nu} , \tag{20}$$

where

$$p^\gamma = P^\gamma + \{{}^{\gamma}{}_{\alpha\beta}\}\left(v^\alpha L^{\beta o} - \rho^{\alpha}{}_{\nu} N^{\beta o\nu}\right). \tag{21}$$

Proof:

First notice that equation (20) follows immediately from the definitions (IV.2.9), (11) and (IV.2.41) of v^{α}, $S^{\gamma\delta}$ and $\rho^{\alpha}{}_{\nu}$. Next expand $(\lambda_{\alpha\delta}{}^{\gamma} - \{^{\gamma}{}_{\alpha\delta}\})$, $\hat{R}_{\alpha\beta}{}^{\gamma}{}_{\delta}$, and $\Gamma^{\gamma}{}_{\beta\alpha}$ about X(t) as a power series in δx^{μ}:

$$(\lambda_{\alpha\delta}{}^{\gamma} - \{^{\gamma}{}_{\alpha\delta}\})\Big|_{x} = (\lambda_{\alpha\delta}{}^{\gamma} - \{^{\gamma}{}_{\alpha\delta}\})\Big|_{X} + \delta x^{\mu}\, \partial_{\mu}(\lambda_{\alpha\delta}{}^{\gamma} - \{^{\gamma}{}_{\alpha\delta}\})\Big|_{X} + \ldots, \tag{22}$$

$$\hat{R}_{\alpha\beta}{}^{\gamma}{}_{\delta}\Big|_{x} = \hat{R}_{\alpha\beta}{}^{\gamma}{}_{\delta}\Big|_{X} + \delta x^{\mu}\, \partial_{\mu}\hat{R}_{\alpha\beta}{}^{\gamma}{}_{\delta}\Big|_{X} + \ldots\,, \tag{23}$$

$$\Gamma^{\gamma}{}_{\beta\alpha}\Big|_{x} = \Gamma^{\gamma}{}_{\beta\alpha}\Big|_{X} + \delta x^{\mu}\, \partial_{\mu}\Gamma^{\gamma}{}_{\beta\alpha}\Big|_{X} + \ldots\,. \tag{24}$$

When these expansions are substituted into equations (14) and (15), the coefficients may be factored out of the integrals leaving the integrals in the form of integrated moments. Since they are negligible, all integrated moments except $M^{\gamma\delta}$, $M^{\beta\gamma\delta}$ and $N^{\gamma\delta\alpha}$ may be dropped. Also, the evaluations at X(t) may be dropped as implicit. The only non-trivial equations are (14) with n = 0,1,2 and (15) with n = 0,1. These are

$$\frac{d}{dt} M^{\gamma o} = (\lambda_{\alpha\delta}{}^{\gamma} - \{^{\gamma}{}_{\alpha\delta}\})M^{\alpha\delta} + \frac{1}{2}\hat{R}_{\alpha\beta}{}^{\gamma}{}_{\delta}\, N^{\beta\alpha\delta} + (\partial_{\mu}\lambda_{\alpha\delta}{}^{\gamma} - \partial_{\mu}\{^{\gamma}{}_{\alpha\delta}\})M^{\mu\alpha\delta}, \tag{25}$$

$$\frac{d}{dt} M^{\beta\gamma o} = M^{\gamma\beta} - v^{\beta} M^{\gamma o} + (\lambda_{\alpha\delta}{}^{\gamma} - \{^{\gamma}{}_{\alpha\delta}\})M^{\beta\alpha\delta}, \tag{26}$$

$$0 = M^{\beta\gamma\delta} + M^{\delta\gamma\beta} - v^{\delta} M^{\beta\gamma o} - v^{\beta} M^{\delta\gamma o}, \tag{27}$$

$$\frac{d}{dt} N^{\gamma\delta o} = 2\, M^{[\gamma\delta]} + 2\, \Gamma^{[\gamma}{}_{\beta\alpha}\, N^{\delta]\beta\alpha}\,, \tag{28}$$

$$0 = N^{\gamma\delta\beta} - v^{\beta} N^{\gamma\delta o} + 2\, M^{\beta[\gamma\delta]}\,. \tag{29}$$

The remainder of the proof consists of analyzing these equations in reverse order. Briefly:

(i) Solve (29) for $M^{\beta[\gamma\delta]}$ in terms of $\rho^{\beta}{}_{\nu}\ N^{\gamma\delta\nu}$.

(ii) Solve (28) for $M^{[\gamma\delta]}$ in terms of $S^{\gamma\delta}$ and $\rho^{\beta}{}_{\nu}\ N^{\gamma\delta\nu}$.

(iii) Solve (27) for $M^{\beta(\gamma\delta)}$ in terms of $L^{\gamma\delta}$ and $\rho^{\beta}{}_{\nu}\ N^{\gamma\delta\nu}$.

(iv) Solve (26) for $M^{\gamma\delta}$ in terms of P^{γ}, $L^{\gamma\delta}$ and $\rho^{\beta}{}_{\nu}\ N^{\gamma\delta\nu}$, and equate the antisymmetric part to $M^{[\gamma\delta]}$ found in (ii) to obtain the equation of motion for $L^{\gamma\delta} + S^{\gamma\delta}$.

(v) Solve (25) for the equation of motion for P^{γ}.

I now proceed to analyze equations (25) through (29) in more detail.

First, using (20), equation (29) may be written as

$$M^{\beta[\gamma\delta]} = -\frac{1}{2}\rho^{\beta}{}_{\nu}\ N^{\gamma\delta\nu}. \qquad (30)$$

Second, using (20) and

$$\frac{d}{dt} = v^{\alpha}\ \partial_{\alpha}\ , \qquad (31)$$

equation (28) may be written as

$$\begin{aligned} 2\ M^{[\gamma\delta]} &= \frac{d}{dt} S^{\gamma\delta} + v^{\alpha}\ \Gamma^{\gamma}{}_{\beta\alpha}\ S^{\beta\delta} + v^{\alpha}\ \Gamma^{\delta}{}_{\beta\alpha}\ S^{\gamma\beta} - 2\ \Gamma^{[\gamma}{}_{\beta\alpha}\ N^{\delta]\beta\nu}\ \rho^{\alpha}{}_{\nu} \\ &= e_{\hat{\kappa}}{}^{\gamma}\ e_{\hat{\lambda}}{}^{\delta}\ \nabla_{v} S^{\hat{\kappa}\hat{\lambda}} - 2\ \Gamma^{[\gamma}{}_{\beta\alpha}\ N^{\delta]\beta\nu}\ \rho^{\alpha}{}_{\nu}\ . \end{aligned} \qquad (32)$$

(Recall that orthonormal indices are corrected with $\Gamma^{\hat{\kappa}}{}_{\hat{\lambda}\alpha}$.)

Third, cyclic permute $\beta\gamma\delta$ in (27) twice:

$$0 = M^{\gamma\delta\beta} + M^{\beta\delta\gamma} - v^{\beta}\ M^{\gamma\delta o} - v^{\gamma}\ M^{\beta\delta o}, \qquad (33)$$

$$0 = M^{\delta\beta\gamma} + M^{\gamma\beta\delta} - v^{\gamma}\ M^{\delta\beta o} - v^{\delta}\ M^{\gamma\beta o}. \qquad (34)$$

Add (27) and (33) and subtract (34):

$$0 = M^{\beta(\gamma\delta)} - M^{\delta[\beta\gamma]} - M^{\gamma[\beta\delta]} - v^{\beta} M^{(\gamma\delta)o} - v^{\delta} M^{[\beta\gamma]o} - v^{\gamma} M^{[\beta\delta]o} . \tag{35}$$

Notice that in solving for $M^{\beta(\gamma\delta)}$, all terms are known except $M^{(\gamma\delta)o}$. To find an expression for $M^{(\gamma\delta)o}$, use the identity,

$$2 M^{(\gamma\delta)o} = 2 M^{\gamma[\delta o]} + 2 M^{\delta[\gamma o]} + M^{\gamma o\delta} + M^{\delta o\gamma}, \tag{36}$$

which follows by writing out the symmetrizations. Combine this with the equation obtained from (34) by setting $\beta = 0$:

$$2 M^{(\gamma\delta)o} = 2 M^{\gamma[\delta o]} + 2 M^{\delta[\gamma o]} + v^{\gamma} M^{\delta oo} + v^{\delta} M^{\gamma oo}. \tag{37}$$

Then using

$$2 M^{[\gamma o]o} = M^{\gamma oo} - M^{o\gamma o} = M^{\gamma oo}, \tag{38}$$

equation (37) becomes

$$M^{(\gamma\delta)o} = M^{\gamma[\delta o]} + M^{\delta[\gamma o]} + v^{\gamma} M^{[\delta o]o} + v^{\delta} M^{[\gamma o]o}. \tag{39}$$

Hence, equation (35) may be written as

$$\begin{aligned} M^{\beta(\gamma\delta)} &= M^{\gamma[\beta\delta]} + M^{\delta[\beta\gamma]} + v^{\gamma} M^{[\beta\delta]o} + v^{\delta} M^{[\beta\gamma]o} \\ &\quad + v^{\beta}(M^{\gamma[\delta o]} + M^{\delta[\gamma o]} + v^{\gamma} M^{[\delta o]o} + v^{\delta} M^{[\gamma o]o}) \\ &= \rho^{\beta}{}_{\mu}(M^{\gamma[\mu\delta]} + M^{\delta[\mu\gamma]} + v^{\gamma} M^{[\mu\delta]o} + v^{\delta} M^{[\mu\gamma]o}). \end{aligned} \tag{40}$$

It is useful to notice that equation (27) may be rederived from (40) so that all of the information in (27) has been incorporated into (40). Now substitute (30) into (39) and (40) to obtain

$$M^{(\gamma\delta)o} = v^{(\gamma} L^{\delta)o} - \rho^{(\gamma}{}_{\nu} N^{\delta)o\nu}, \tag{41}$$

$$M^{\beta(\gamma\delta)} = \rho^{\beta}{}_{\mu} (-v^{(\gamma} L^{\delta)\mu} + \rho^{(\gamma}{}_{\nu} N^{\delta)\mu\nu}). \tag{42}$$

The sum of (42) and (30) is equation (19).

Fourth, solve equation (26) for $M^{\gamma\delta}$ using (30), (41) and (42):

$$\begin{aligned} M^{\gamma\delta} &= v^{\delta} M^{\gamma o} + \frac{d}{dt} M^{\delta\gamma o} - \lambda_{\alpha\beta}{}^{\gamma} M^{\delta[\alpha\beta]} + \{^{\gamma}{}_{\alpha\beta}\} M^{\delta(\alpha\beta)} \\ &= v^{\delta} P^{\gamma} + \frac{d}{dt} (\frac{1}{2} L^{\delta\gamma} + v^{(\delta} L^{\gamma)o} - \rho^{(\delta}{}_{\nu} N^{\gamma)o\nu}) \\ &\quad + \frac{1}{2} \lambda_{\alpha\beta}{}^{\gamma} \rho^{\delta}{}_{\nu} N^{\alpha\beta\nu} + \{^{\gamma}{}_{\alpha\beta}\} \rho^{\delta}{}_{\mu} (- v^{\alpha} L^{\beta\mu} + \rho^{\alpha}{}_{\nu} N^{\beta\mu\nu}). \end{aligned} \tag{43}$$

This is equation (18). To find the equation of motion for $L^{\gamma\delta} + S^{\gamma\delta}$, equate equation (32) to twice the antisymmetric part of (43):

$$\begin{aligned} & e_{\hat{\kappa}}{}^{\gamma} e_{\hat{\lambda}}{}^{\delta} \nabla_{\mathbf{v}} S^{\hat{\kappa}\hat{\lambda}} - 2 \Gamma^{[\gamma}{}_{\beta\alpha} N^{\delta]\beta\nu} \rho^{\alpha}{}_{\nu} \\ &= 2 v^{[\delta} P^{\gamma]} + \frac{d}{dt} L^{\delta\gamma} + \lambda_{\alpha\beta}{}^{[\gamma} \rho^{\delta]}{}_{\nu} N^{\alpha\beta\nu} \\ &\quad -2 v^{[\delta} \{^{\gamma]}{}_{\alpha\beta}\} (- v^{\alpha} L^{\beta o} + \rho^{\alpha}{}_{\nu} N^{\beta o\nu}) \\ &\quad + \{^{\gamma}{}_{\alpha\beta}\} (- v^{\alpha} L^{\beta\delta} + \rho^{\alpha}{}_{\nu} N^{\beta\delta\nu}) \\ &\quad - \{^{\delta}{}_{\alpha\beta}\} (- v^{\alpha} L^{\beta\gamma} + \rho^{\alpha}{}_{\nu} N^{\beta\gamma\nu}) \\ &= 2 v^{[\delta} p^{\gamma]} + \nabla_{\mathbf{v}} L^{\delta\gamma} + \lambda_{\alpha\beta}{}^{[\gamma} \rho^{\delta]}{}_{\nu} N^{\alpha\beta\nu} \\ &\quad - 2 \{^{[\gamma}{}_{\alpha\beta}\} N^{\delta]\beta\nu} \rho^{\alpha}{}_{\nu}, \end{aligned} \tag{44}$$

where p^γ is defined by equation (21). Upon rearrangement, equation (44) becomes (17).

Finally, the equation of motion for P^γ may be found by substituting (20), (30), (32), (42) and (43) into (25):

$$\frac{d}{dt} P^\gamma = \lambda_{\alpha\delta}{}^\gamma(\frac{1}{2} e_{\hat\kappa}{}^\alpha e_{\hat\lambda}{}^\delta \nabla_v S^{\hat\kappa\hat\lambda} - \Gamma^\alpha{}_{\mu\nu} N^{\delta\mu\beta} \rho^\nu{}_\beta)$$

$$- \{^\gamma{}_{\alpha\delta}\}[v^\alpha P^\delta + \frac{d}{dt} (v^\alpha L^{\delta o} - \rho^\alpha{}_\nu N^{\delta o\nu}) + \frac{1}{2} \lambda_{\mu\nu}{}^\alpha \rho^\delta{}_\beta N^{\mu\nu\beta}$$

$$+ \{^\delta{}_{\mu\nu}\}\rho^\alpha{}_\varepsilon(-v^\mu L^{\nu\varepsilon} + \rho^\mu{}_\beta N^{\nu\varepsilon\beta})]$$

$$+ \frac{1}{2} \hat R_{\alpha\beta}{}^\gamma{}_\delta(v^\delta S^{\beta\alpha} + \rho^\delta{}_\nu N^{\beta\alpha\nu})$$

$$+ (\partial_\mu\lambda_{\alpha\delta}{}^\gamma)(- \frac{1}{2} \rho^\mu{}_\nu N^{\alpha\delta\nu})$$

$$- (\partial_\mu\{^\gamma{}_{\alpha\delta}\})\rho^\mu{}_\varepsilon(- v^\alpha L^{\delta\varepsilon} + \rho^\alpha{}_\nu N^{\delta\varepsilon\nu}). \tag{45}$$

Now it helps to know the answer and compute (using (21) and the definition of $\tilde R_{\alpha\beta}{}^\gamma{}_\delta$ in the first step, (31) in the second step and (45) in the third step):

$$\nabla_v p^\gamma - \frac{1}{2} L^{\beta\alpha} \tilde R_{\alpha\beta}{}^\gamma{}_\delta v^\delta - \frac{1}{2} S^{\beta\alpha} \hat R_{\alpha\beta}{}^\gamma{}_\delta v^\delta$$

$$= \frac{d}{dt} [P^\gamma + \{^\gamma{}_{\alpha\beta}\}(v^\alpha L^{\beta o} - \rho^\alpha{}_\nu N^{\beta o\nu})]$$

$$+ v^\alpha\{^\gamma{}_{\beta\alpha}\}[P^\beta + \{^\beta{}_{\mu\nu}\}(v^\mu L^{\nu o} - \rho^\mu{}_\delta N^{\nu o\delta})]$$

$$- v^\delta L^{\beta\alpha}(\partial_\alpha\{^\gamma{}_{\delta\beta}\} + \{^\gamma{}_{\varepsilon\alpha}\}\{^\varepsilon{}_{\delta\beta}\}) - \frac{1}{2} S^{\beta\alpha} \hat R_{\alpha\beta}{}^\gamma{}_\delta v^\delta$$

$$= \frac{d}{dt} P^\gamma - \frac{1}{2} S^{\beta\alpha} \hat R_{\alpha\beta}{}^\gamma{}_\delta v^\delta$$

$$+ \{^\gamma{}_{\alpha\beta}\}[v^\alpha P^\beta + \frac{d}{dt} (v^\alpha L^{\beta o} - \rho^\alpha{}_\nu N^{\beta o\nu})$$

$$+ \{^\beta{}_{\mu\nu}\}v^\alpha(v^\mu L^{\nu o} - \rho^\mu{}_\delta N^{\nu o\delta}) - \{^\beta{}_{\mu\nu}\} v^\mu L^{\nu\alpha}]$$

$$+ (\partial_\mu\{^\gamma{}_{\alpha\beta}\}) v^\mu (v^\alpha L^{\beta o} - \rho^\alpha{}_\nu N^{\beta o\nu}) - (\partial_\mu\{^\gamma{}_{\alpha\beta}\}) v^\alpha L^{\beta\mu}$$

$$= \lambda_{\alpha\beta}{}^{\gamma}(\frac{1}{2} e_{\hat{\kappa}}{}^{\alpha} e_{\hat{\lambda}}{}^{\beta} \nabla_v S^{\hat{\kappa}\hat{\lambda}} - \Gamma^{\alpha}{}_{\mu\nu} N^{\beta\mu\delta} \rho^{\nu}{}_{\delta})$$

$$- \{^{\gamma}{}_{\alpha\beta}\}(\frac{1}{2} \lambda_{\mu\nu}{}^{\alpha} \rho^{\beta}{}_{\delta} N^{\mu\nu\delta} + \{^{\beta}{}_{\mu\nu}\}\rho^{\mu}{}_{\delta} N^{\nu\alpha\delta})$$

$$+ \frac{1}{2} \hat{R}_{\alpha\beta}{}^{\gamma}{}_{\delta} \rho^{\delta}{}_{\nu} N^{\beta\alpha\nu} - (\partial_{\mu}\lambda_{\alpha\beta}{}^{\gamma})\frac{1}{2} \rho^{\mu}{}_{\nu} N^{\alpha\beta\nu}$$

$$- (\partial_{\mu}\{^{\gamma}{}_{\alpha\beta}\})\rho^{\alpha}{}_{\nu} N^{\beta\mu\nu}$$

$$= \frac{1}{2} \lambda_{\alpha\beta}{}^{\gamma} e_{\hat{\kappa}}{}^{\alpha} e_{\hat{\lambda}}{}^{\beta} \nabla_v S^{\hat{\kappa}\hat{\lambda}}$$

$$+ \frac{1}{2} \rho^{\delta}{}_{\nu} N^{\beta\alpha\nu}(\hat{R}_{\alpha\beta}{}^{\gamma}{}_{\delta} - 2\partial_{\alpha}\{^{\gamma}{}_{\delta\beta}\} - 2\{^{\gamma}{}_{\varepsilon\alpha}\}\{^{\varepsilon}{}_{\delta\beta}\}$$

$$+ \partial_{\delta}\lambda_{\alpha\beta}{}^{\gamma} + \{^{\gamma}{}_{\varepsilon\delta}\}\lambda_{\alpha\beta}{}^{\varepsilon} - 2\Gamma^{\varepsilon}{}_{\alpha\delta} \lambda_{\varepsilon\beta}{}^{\gamma})$$

$$= \frac{1}{2} \lambda_{\alpha\beta}{}^{\gamma} e_{\hat{\kappa}}{}^{\alpha} e_{\hat{\lambda}}{}^{\beta} \nabla_v S^{\hat{\kappa}\hat{\lambda}} + \frac{1}{2} \rho^{\delta}{}_{\nu} N^{\beta\alpha\nu} g^{\gamma\varepsilon} \nabla_{\varepsilon}\lambda_{\alpha\beta\delta} , \qquad (46)$$

where the last step uses the identity,

$$\hat{R}_{\alpha\beta}{}^{\gamma}{}_{\delta} - \tilde{R}_{\alpha\beta}{}^{\gamma}{}_{\delta} = g^{\gamma\varepsilon}\nabla_{\varepsilon}\lambda_{\alpha\beta\delta} - \nabla_{\delta}\lambda_{\alpha\beta}{}^{\gamma} + \lambda_{\alpha\varepsilon}{}^{\gamma} \lambda^{\varepsilon}{}_{\beta\delta} - \lambda_{\alpha\varepsilon\delta} \lambda^{\varepsilon}{}_{\beta}{}^{\gamma} . \qquad (47)$$

Equation (46) coincides with (16) and completes the proof of the theorem.

Q.E.D.

4. Discussion and Special Cases

In this section, I discuss Theorem IV.2 and the special cases when the orbital or spin angular momenta or both are negligible. I start by comparing these results with the corresponding results for metric theories. Since the metric theories are just the metric-connection theories in which the connection is metric-compatible and torsion-free, I regain the equations for the metric theories by setting the torsion to zero in the equations for the metric-Cartan connection theories.

The metric theory results have been obtained and discussed by many authors, including Mathisson [1937], Papapetrou [1951], Dixon [1970a,b, 1973, 1974], Madore [1966, 1969], and Bieglbock [1967]. I wish to emphasize that when these authors refer to "spin angular momentum," they do not mean the elementary particle spin. Rather, they mean the total angular momentum, $\tilde{J}^{\gamma\delta} = \int (\delta x^{\gamma} \tilde{T}^{\delta o} - \delta x^{\delta} \tilde{T}^{\gamma o}) \sqrt{-\tilde{g}}\, d^3x$, computed using the metric energy-momentum tensor, $\tilde{T}^{\gamma\delta}$. (Notice that $\tilde{J}^{\gamma\delta} = L^{\gamma\delta} + S^{\gamma\delta}$, to lowest order in δx^{γ}.) They denote the quantity, $\tilde{J}^{\gamma\delta}$, by $S^{\gamma\delta}$, whereas I reserve the symbol, $S^{\gamma\delta}$, for the integrated elementary particle spin. Further, they talk about $\tilde{J}^{\gamma\delta}$ as if it were just the orbital angular momentum, $L^{\gamma\delta}$, computed about the "center of mass." Since the above mentioned authors do not distinguish between $L^{\gamma\delta}$, $S^{\gamma\delta}$, and $L^{\gamma\delta} + S^{\gamma\delta}$, they do not obtain the metric theory results presented below which show that these three quantities behave identically in the metric theories. On the other hand, the new metric theory results may be obtained from the old results by replacing $\tilde{J}^{\gamma\delta}$ by $L^{\gamma\delta}$, $S^{\gamma\delta}$, or $L^{\gamma\delta} + S^{\gamma\delta}$.

I begin with the simplest situation in which the body has neither spin nor orbital angular momentum; i.e. a gravitational monopole. In the metric-Cartan connection case we have:

Corollary IV.1:

In a metric-Cartan connection theory of gravity, if the integrated moments, $M^{\beta_1 \ldots \beta_n \gamma\delta}$ *with* $n \geq 1$ *and* $N^{\beta_1 \ldots \beta_n \gamma\delta\alpha}$ *with* $n \geq 0$ *, are negligible compared to* $M^{\gamma\delta}$ *, then*

$$\nabla_v P^\gamma = 0 \,, \tag{1}$$

$$P^\gamma = m\, v^\gamma \,, \tag{2}$$

$$M^{\gamma\delta} = P^\gamma v^\delta \,, \tag{3}$$

where

$$m^2 = s\, P^\gamma P_\gamma \quad . \tag{4}$$

Further, the curve, X(t)*, is a geodesic of the Christoffel connection and*

$$\nabla_v m = 0. \tag{5}$$

Proof:

Setting $M^{\beta\gamma\delta}$, $N^{\gamma\delta\alpha}$, $L^{\gamma\delta}$ and $S^{\gamma\delta}$ equal to zero in equations (IV.3.16-21), they become equations (1), (3) and

$$v^{[\gamma} P^{\delta]} = 0 \,. \tag{6}$$

Equation (6) says that P^γ and v^γ are linearly dependent. Since $v^\gamma \neq 0$, there exists a function, m(t), such that (2) is satisfied. Recall that

v^γ is a timelike unit vector; so that $v^\gamma v_\gamma = s = \pm 1$, where the signature of the metric is $(s, -s, -s, -s)$. Hence, (4) is the square of (2), Taking the covariant derivative of (4) in the direction v^γ and using (1) yields (5). Finally, substituting (2) into (1) and using (5) gives, $\nabla_v v^\gamma = 0$, which says that X(t) is a geodesic of the Christoffel connection.

Q.E.D.

The metric theory results are obtained by setting the torsion to zero, but the torsion does not appear in the equations of Corollary IV.1. Hence:

Corollary IV.2:

In a metric theory of gravity, if the integrated moments, $M^{\beta_1\ldots\beta_n\gamma\delta}$ *with* $n \geq 1$ *and* $N^{\beta_1\ldots\beta_n\gamma\delta\alpha}$ *with* $n \geq 0$ *, are negligible compared to* $M^{\gamma\delta}$ *, then*

$$\nabla_v P^\gamma = 0 \, , \tag{7}$$

$$P^\gamma = m \, v^\gamma \, , \tag{8}$$

$$M^{\gamma\delta} = P^\gamma \, v^\delta \, , \tag{9}$$

where

$$m^2 = s \, P^\gamma \, P_\gamma \, . \tag{10}$$

Further, the curve, X(t) *, is a geodesic and*

$$\nabla_v m = 0 \, . \tag{11}$$

Corollaries IV.1 and IV.2 show that in a metric-Cartan connection theory of gravity, a body with no spin or orbital angular momentum does not feel the torsion and moves in the same manner (i.e. along a Christoffel geodesic) as it would in a metric theory provided the metric is the same. A similar conclusion holds in the slightly more general situation when the body has orbital angular momentum but no spin angular momentum:

Corollary IV.3:

In a metric-Cartan connection theory of gravity, if the integrated moments, $M^{\beta_1\ldots\beta_n\gamma\delta}$ *with* $n \geq 2$ *and* $N^{\beta_1\ldots\beta_n\gamma\delta\alpha}$ *with* $n \geq 0$ *, are negligible compared to* $M^{\gamma\delta}$ *and* $M^{\beta\gamma\delta}$ *, then*

$$\nabla_v \mathcal{P}^\gamma = \frac{1}{2} L^{\beta\alpha} \tilde{R}_{\alpha\beta}{}^{\gamma}{}_{\delta}\, v^\delta, \tag{12}$$

$$\nabla_v L^{\gamma\delta} = -\,2\, v^{[\gamma} \mathcal{P}^{\delta]}, \tag{13}$$

$$M^{\gamma\delta} = v^{(\gamma} P^{\delta)} + \frac{d}{dt}\left(v^{(\gamma} L^{\delta)o}\right) - \{^{(\gamma}{}_{\alpha\beta}\}\rho^{\delta)}{}_{\mu}\, v^\alpha L^{\beta\mu}, \tag{14}$$

$$M^{\beta\gamma\delta} = -\,\rho^{\beta}{}_{\mu} v^{(\gamma} L^{\delta)\mu}, \tag{15}$$

where

$$\mathcal{P}^\gamma = P^\gamma + \{^{\gamma}{}_{\alpha\beta}\} v^\alpha L^{\beta o}. \tag{16}$$

Proof:

Setting $N^{\gamma\delta\alpha}$ and $S^{\gamma\delta}$ equal to zero in equations (IV.3.16-21), they become equations (12), (13), (15), (16), and

$$M^{\gamma\delta} = v^\delta P^\gamma + \frac{d}{dt}\left(\frac{1}{2} L^{\delta\gamma} + v^{(\delta} L^{\gamma)o}\right) - \{^{\gamma}{}_{\alpha\beta}\}\rho^{\delta}{}_{\mu}\, v^\alpha L^{\beta\mu}. \tag{17}$$

Using (13), the antisymmetric part of (17) says that $M^{[\gamma\delta]} = 0$. Hence, the symmetric part of (17) becomes (14).

Q.E.D.

As in Corollary IV.1 the equations in Corollary IV.3 contain no torsion. Hence, the metric theory results are immediate:

Corollary IV.4:

In a metric theory of gravity, if the integrated moments, $M^{\beta_1 \ldots \beta_n \gamma\delta}$ *with* $n \geq 2$ *and* $N^{\beta_1 \ldots \beta_n \gamma\delta\alpha}$ *with* $n \geq 0$ *, are negligible compared to* $M^{\gamma\delta}$ *and* $M^{\beta\gamma\delta}$ *, then*

$$\nabla_v \mathcal{P}^\gamma = \frac{1}{2} L^{\beta\alpha} \tilde{R}_{\alpha\beta}{}^{\gamma}{}_{\delta} v^\delta , \tag{18}$$

$$\nabla_v L^{\gamma\delta} = - 2 v^{[\gamma} \mathcal{P}^{\delta]}, \tag{19}$$

$$M^{\gamma\delta} = v^{(\gamma} P^{\delta)} + \frac{d}{dt} (v^{(\gamma} L^{\delta)o}) - \{^{(\gamma}{}_{\alpha\beta}\} \rho^{\delta)}{}_{\mu} v^\alpha L^{\beta\mu}, \tag{20}$$

$$M^{\beta\gamma\delta} = - \rho^\beta{}_\mu v^{(\gamma} L^{\delta)\mu}, \tag{21}$$

where

$$\mathcal{P}^\gamma = P^\gamma + \{^\gamma{}_{\alpha\beta}\} v^\alpha L^{\beta o}. \tag{22}$$

Corollaries IV.3 and IV.4 show that in a metric-Cartan connection theory of gravity, a body, whose integrated elementary particle spin is negligible, does not feel the torsion and propagates its momentum and orbital angular momentum in the same manner as it would in a metric theory provided the metric is the same. Note that Corollaries IV.3 and IV.4 are appropriate to photons for which the standard Lagrangian, $L_{em} = - F_{\mu\nu} F^{\mu\nu}$, yields $S^\beta{}_\alpha{}^a = 0$.

The situation changes drastically when the spin is non-zero. The simplest situation is when the orbital angular momentum is zero:

Corollary IV.5:

In a metric-Cartan connection theory of gravity, if the integrated moments, $M^{\beta_1 \ldots \beta_n \gamma\delta}$ *with* $n \geq 1$ *and* $N^{\beta_1 \ldots \beta_n \gamma\delta\alpha}$ *with* $n \geq 1$ *, are negligible compared to* $M^{\gamma\delta}$ *and* $N^{\gamma\delta\alpha}$ *, then*

$$\nabla_v P^\gamma = \frac{1}{2} S^{\beta\alpha} \hat{R}_{\alpha\beta}{}^{\gamma}{}_{\delta} v^\delta + P^\beta \lambda_{\beta\delta}{}^{\gamma} v^\delta , \tag{23}$$

$$e_{\hat{\kappa}}{}^{\gamma} e_{\hat{\lambda}}{}^{\delta} \nabla_v S^{\hat{\kappa}\hat{\lambda}} = - 2 v^{[\gamma} P^{\delta]}, \tag{24}$$

$$M^{\gamma\delta} = v^\delta P^\gamma , \tag{25}$$

$$N^{\gamma\delta\alpha} = v^\alpha S^{\gamma\delta} . \tag{26}$$

Proof:

Setting $M^{\beta\gamma\delta}$ and $L^{\gamma\delta}$ equal to zero in equation (IV.3.19) yields

$$0 = - \rho^{\beta}{}_{\mu} (\frac{1}{2} N^{\gamma\delta\mu} - \rho^{(\gamma}{}_{\nu} N^{\delta)\mu\nu}), \tag{27}$$

whose antisymmetric part on γ and δ is

$$\rho^{\beta}{}_{\mu} N^{\gamma\delta\mu} = 0. \tag{28}$$

Hence, setting $M^{\beta\gamma\delta}$, $L^{\gamma\delta}$ and $\rho^{\beta}{}_{\mu} N^{\gamma\delta\mu}$ equal to zero in equations (IV.3.16,17,18,20, and 21) yields equations (24), (25), (26), and

$$\nabla_v P^\gamma = \frac{1}{2} S^{\beta\alpha} \hat{R}_{\alpha\beta}{}^{\gamma}{}_{\delta} v^\delta + \frac{1}{2} \lambda_{\alpha\beta}{}^{\gamma} e_{\hat{\kappa}}{}^{\alpha} e_{\hat{\lambda}}{}^{\beta} \nabla_v S^{\hat{\kappa}\hat{\lambda}}, \tag{29}$$

which becomes (23) upon substitution of (24).

Q.E.D.

Setting the torsion to zero in the equations of Corollary IV.5 yields:

Corollary IV.6:

In a metric theory of gravity, if the integrated moments, $M^{\beta_1 \ldots \beta_n \gamma\delta}$ *with* $n \geq 1$ *and* $N^{\beta_1 \ldots \beta_n \gamma\delta\alpha}$ *with* $n \geq 1$ *, are negligible compared to* $M^{\gamma\delta}$ *and* $N^{\gamma\delta\alpha}$*, then*

$$\nabla_v P^\gamma = \frac{1}{2} S^{\beta\alpha} \tilde{R}_{\alpha\beta}{}^{\gamma}{}_{\delta} v^\delta, \tag{30}$$

$$\nabla_v S^{\gamma\delta} = -2 v^{[\gamma} P^{\delta]}, \tag{31}$$

$$M^{\gamma\delta} = v^\delta P^\gamma, \tag{32}$$

$$N^{\gamma\delta\alpha} = v^\alpha S^{\gamma\delta}. \tag{33}$$

A comparison of Corollaries IV.4 and IV.6 shows that in a metric theory of gravity, the propagation of the momentum and angular momentum of a body is independent of whether its angular momentum is all orbital or all spin. However, in the presence of torsion, Corollary IV.3 shows that a body with only orbital angular momentum will ignore the torsion; whereas Corollary IV.5 shows that a body with only spin angular momentum will feel the torsion in three ways:

(i) There is a spin-torsion coupling in equation (24) but no orbital-torsion coupling in equation (13). The spin-torsion coupling may be described by saying that in (24) the spin propagates according to the Cartan connection (rather than the Christoffel connection) with a torque $-2 v^{[\gamma} P^{\delta]}$. Alternatively, rewriting (24) as

$$\nabla_v S^{\gamma\delta} = -2 v^{[\gamma} P^{\delta]} + 2 S^{\beta[\gamma} \lambda^{\delta]}{}_{\beta\alpha} v^\alpha, \tag{34}$$

the spin propagates according to the Christoffel connection but has an additional spin-torsion torque.

(ii) The spin-curvature force in (23) couples the spin to the Cartan curvature whereas the orbital-curvature force in (12) couples the orbital angular momentum to only the Christoffel curvature.

(iii) Equation (23) contains a momentum-torsion force not present in equation (12). The presence of this force when and only when the spin is non-zero, is perplexing from a physical point of view. Mathematically, it arises in equation (29) as a coupling between the torsion and the derivative of the spin.

It is informative to rewrite equation (23) using Cartan covariant derivatives:

$$e_{\hat{\kappa}}{}^{\gamma} \, \nabla_v P^{\hat{\kappa}} = \frac{1}{2} S^{\beta\alpha} \, \hat{R}_{\alpha\beta}{}^{\gamma}{}_{\delta} \, v^{\delta} + P^{\beta} \, Q_{\beta}{}^{\gamma}{}_{\delta} \, v^{\delta} . \tag{35}$$

In this form equations (35) and (24) bear a close resemblance to the original differential conservation laws (IV.3.1) and (IV.3.2). So perhaps the correct questions are: Why is a momentum-torsion force missing in equation (12)? Why do equations (12) and (13) involve Christoffel covariant derivatives and Christoffel curvatures rather than Cartan ones? However, after some manipulation, equations (12) and (13) may be rewritten as,

$$e_{\hat{\kappa}}{}^{\gamma} \, \nabla_v \overline{\mathcal{P}}^{\hat{\kappa}} = \frac{1}{2} L^{\beta\alpha} \, \hat{R}_{\alpha\beta}{}^{\gamma}{}_{\delta} \, v^{\delta} + \overline{\mathcal{P}}^{\beta} \, Q_{\beta}{}^{\gamma}{}_{\delta} \, v^{\delta} + \frac{1}{2} L^{\hat{\mu}\hat{\nu}} \, v^{\hat{\kappa}} \, \nabla^{\gamma} \lambda_{\hat{\mu}\hat{\nu}\hat{\kappa}} , \tag{36}$$

$$e_{\hat{\kappa}}{}^{\gamma} \, e_{\hat{\lambda}}{}^{\delta} \, \nabla_v L^{\hat{\kappa}\hat{\lambda}} = - 2 \, v^{[\gamma} \, \overline{\mathcal{P}}^{\delta]} - v^{[\gamma} \, \lambda_{\alpha\beta}{}^{\delta]} L^{\beta\alpha} - 2 \, v^{\alpha} \, \lambda^{[\gamma}{}_{\beta\alpha} \, L^{\delta]\beta} , \tag{37}$$

where

$$\begin{aligned} \overline{\mathcal{P}}^{\gamma} &= \mathcal{P}^{\gamma} - \frac{1}{2} L^{\beta\alpha} \, \lambda_{\alpha\beta}{}^{\gamma} \\ &= P^{\gamma} + \{{}^{\gamma}{}_{\alpha\beta}\} v^{\alpha} \, L^{\beta o} - \frac{1}{2} L^{\beta\alpha} \, \lambda_{\alpha\beta}{}^{\gamma} . \end{aligned} \tag{38}$$

In this form there are Cartan covariant derivatives, orbital-Cartan curvature forces, and momentum-torsion forces, but there are also orbital-derivative-of-torsion forces and orbital-torsion torques. Perhaps there is some other redefinition of the momentum and/or the orbital or spin angular momenta which will make the equations of Corollaries IV.3 and IV.5 look identical. I have not been able to find such a redefinition and do not know whether one exists. Hence in their present form, I interpret Corollaries IV.3 and IV.5 as saying that in the presence of torsion, spin and orbital angular momentum do not behave in the same way.

This brings us to the most complicated situation in which the spin and orbital angular momenta are both non-zero. The metric theory result is obtained by setting the torsion to zero in Theorem IV.2:

Corollary IV.7:

In a metric theory of gravity, if the integrated moments, $M^{\beta_1\cdots\beta_n\gamma\delta}$ *with* $n \geq 2$ *and* $N^{\beta_1\cdots\beta_n\gamma\delta\alpha}$ *with* $n \geq 1$ *, are negligible compared to* $M^{\gamma\delta}$, $M^{\beta\gamma\delta}$ *and* $N^{\gamma\delta\alpha}$, *then*

$$\nabla_v \mathcal{P}^\gamma = \frac{1}{2}(L^{\beta\alpha} + S^{\beta\alpha})\tilde{R}_{\alpha\beta}{}^{\gamma}{}_{\delta}\, v^\delta \ , \tag{39}$$

$$\nabla_v (L^{\gamma\delta} + S^{\gamma\delta}) = -\,2\, v^{[\gamma}\mathcal{P}^{\delta]}, \tag{40}$$

$$M^{\gamma\delta} = v^\delta\, P^\gamma + \frac{d}{dt}\left(\frac{1}{2} L^{\delta\gamma} + v^{(\gamma} L^{\delta)o} - \rho^{(\gamma}{}_{\nu} N^{\delta)o\nu}\right) - \{{}^{\gamma}{}_{\alpha\beta}\}\rho^{\delta}{}_{\mu}(v^\alpha L^{\beta\mu} - \rho^{\alpha}{}_{\nu} N^{\beta\mu\nu}), \tag{41}$$

$$M^{\beta\gamma\delta} = -\,\rho^{\beta}{}_{\mu}\left(\frac{1}{2} N^{\gamma\delta\mu} + v^{(\gamma} L^{\delta)\mu} - \rho^{(\gamma}{}_{\nu} N^{\delta)\mu\nu}\right), \tag{42}$$

$$N^{\gamma\delta\alpha} = v^\alpha\, S^{\gamma\delta} + \rho^{\alpha}{}_{\nu} N^{\gamma\delta\nu}, \tag{43}$$

where

$$\mathcal{P}^\gamma = P^\gamma + \{{}^{\gamma}{}_{\alpha\beta}\}(v^\alpha L^{\beta o} - \rho^{\alpha}{}_{\nu} N^{\beta o\nu}). \tag{44}$$

Corollary IV.7 generalizes Corollaries IV.4 and IV.6 , showing that in a metric theory of gravity, the propagation of the momentum and total angular momentum of a body is independent of what fraction of the total angular momentum in spin and what fraction is orbital. However, Corollary IV.7 gives no information about how the total angular momentum is divided between spin and orbital, nor about how this division changes with time. For that one must investigate the internal dynamics of the body.

In order to discuss the propagation of the momentum and angular momentum of a body with both spin and orbital angular momenta moving in a gravitational field with torsion, one must investigate the full Theorem IV.2. There is a major difficulty with the propagation equations in Theorem IV.2, which does not occur in Corollaries IV.1 through IV.7. In the corollaries the propagation equations are deterministic once the background geometry is specified. For example in Corollary IV.5 if P^{γ} and $S^{\gamma\delta}$ are specified at an initial time then they are determined at all times. Similarly for P^{γ} in Corollary IV.1, for $\mathcal{P}^{\gamma}$ and $L^{\gamma\delta}$ in Corollary IV.3, and for $\mathcal{P}^{\gamma}$ and $L^{\gamma\delta} + S^{\gamma\delta}$ in Corollary IV.7. In Corollaries IV.1 through IV.6 it is even possible to determine the auxilliary variables $M^{\gamma\delta}$, $M^{\beta\gamma\delta}$ and $N^{\gamma\delta\alpha}$ as appropriate. However, in Corollary IV.7 it is not possible to completely determine $M^{\gamma\delta}$, $M^{\beta\gamma\delta}$ and $N^{\gamma\delta\alpha}$ because the quantities $L^{\gamma\delta}$ and $\rho^{\alpha}{}_{\nu}\, N^{\gamma\delta\nu}$ are neither expressed in terms of $\mathcal{P}^{\gamma}$ and $L^{\gamma\delta} + S^{\gamma\delta}$ nor given their own evolution equations. As discussed above, the splitting of the total angular momentum, $L^{\gamma\delta} + S^{\gamma\delta}$ into $L^{\gamma\delta}$ and $S^{\gamma\delta}$ depends on the internal dynamics of the body. Presumably, the same is true for the value of $\rho^{\alpha}{}_{\nu}\, N^{\gamma\delta\nu}$.

In Theorem IV.2 not only are $M^{\gamma\delta}$, $M^{\beta\gamma\delta}$ and $N^{\gamma\delta\alpha}$ left undetermined, but even the propagation equations, (IV.3.16) and (IV.3.17), are not deterministic. It is not surprising that there are no separate equations for $L^{\gamma\delta}$, $S^{\gamma\delta}$ and $\rho^{\alpha}{}_{\nu}\, N^{\gamma\delta\nu}$ since that would require knowledge of the internal structure. However, that the propagation equations do not determine $\mathcal{P}^{\gamma}$ and $L^{\gamma\delta} + S^{\gamma\delta}$ from initial data is surprising. The reasons are that (1) the torsion does not couple symmetrically to $L^{\gamma\delta}$ and $S^{\gamma\delta}$ (only to $S^{\gamma\delta}$); and (2) there are new couplings between the torsion and $\rho^{\alpha}{}_{\nu}\, N^{\gamma\delta\nu}$ (which vanished in all previous cases). The obvious solution would be to find a new combination of variables for which the propagation equations are deterministic. I have not been able to do so and do not know whether it is possible.

In attempting to make the propagation equations, (IV.3.16) and (IV.3.17), deterministic, it is instructive to rewrite them using all Cartan covariant derivatives and Cartan curvatures:

$$e_{\hat{\kappa}}{}^{\gamma} \nabla_{v} \bar{p}^{\hat{\kappa}} = \frac{1}{2} (L^{\beta\alpha} + S^{\beta\alpha}) \hat{R}_{\alpha\beta}{}^{\gamma}{}_{\delta} v^{\delta} + \bar{p}^{\beta} Q_{\beta}{}^{\gamma}{}_{\delta} v^{\delta}$$

$$- \frac{1}{2} (v^{\hat{\kappa}} L^{\hat{\nu}\hat{\mu}} - \rho^{\hat{\kappa}}{}_{\hat{\lambda}} N^{\hat{\nu}\hat{\mu}\hat{\lambda}}) \nabla^{\gamma} \lambda_{\hat{\mu}\hat{\nu}\hat{\kappa}} , \tag{45}$$

$$e_{\hat{\kappa}}{}^{\gamma} e_{\hat{\lambda}}{}^{\delta} \nabla_{v} (L^{\hat{\kappa}\hat{\lambda}} + S^{\hat{\kappa}\hat{\lambda}}) = - 2 v^{[\gamma} \bar{p}^{\delta]}$$

$$- 2 \lambda^{[\gamma}{}_{\beta\alpha} (L^{\delta]\beta} v^{\alpha} - N^{\delta]\beta\nu} \rho^{\alpha}{}_{\nu}) + \lambda_{\alpha\beta}{}^{[\gamma} (v^{\delta]} L^{\beta\alpha} - \rho^{\delta]}{}_{\nu} N^{\beta\alpha\nu}), \tag{46}$$

where

$$\bar{p}^{\gamma} = p^{\gamma} - \frac{1}{2} L^{\beta\alpha} \lambda_{\alpha\beta}{}^{\gamma}$$

$$= P^{\gamma} + \{{}^{\gamma}{}_{\alpha\beta}\} (v^{\alpha} L^{\beta o} - \rho^{\alpha}{}_{\nu} N^{\beta o\nu}) - \frac{1}{2} L^{\beta\alpha} \lambda_{\alpha\beta}{}^{\gamma} . \tag{47}$$

Alternatively, equations (IV.3.16) and (IV.3.17) may be rewritten using all Christoffel covariant derivatives and Christoffel curvatures:

$$\nabla_{v} \bar{\bar{p}}^{\gamma} = \frac{1}{2} (L^{\beta\alpha} + S^{\beta\alpha}) \tilde{R}_{\alpha\beta}{}^{\gamma}{}_{\delta} v^{\delta}$$

$$+ \frac{1}{2} (v^{\delta} S^{\beta\alpha} + \rho^{\delta}{}_{\nu} N^{\beta\alpha\nu}) \nabla^{\gamma} \lambda_{\alpha\beta\delta} , \tag{48}$$

$$\nabla_{v} (L^{\gamma\delta} + S^{\gamma\delta}) = - 2 v^{[\gamma} \bar{\bar{p}}^{\delta]}$$

$$+ 2 \lambda^{[\gamma}{}_{\beta\alpha} (S^{\delta]\beta} v^{\alpha} + N^{\delta]\beta\nu} \rho^{\alpha}{}_{\nu}) - \lambda_{\alpha\beta}{}^{[\gamma} (v^{\delta]} S^{\beta\alpha} + \rho^{\delta]}{}_{\nu} N^{\beta\alpha\nu}), \tag{49}$$

where

$$\overline{\overline{\mathcal{P}}}^{\gamma} = \mathcal{P}^{\gamma} + \frac{1}{2} S^{\beta\alpha} \lambda_{\alpha\beta}{}^{\gamma}$$

$$= P^{\gamma} + \{{}^{\gamma}_{\alpha\beta}\}(v^{\alpha} L^{\beta o} - \rho^{\alpha}{}_{\nu} N^{\beta o \nu}) + \frac{1}{2} S^{\beta\alpha} \lambda_{\alpha\beta}{}^{\gamma} \quad . \qquad (50)$$

Neither of these forms of the propagation equations, (45) and (46) nor (48) and (49), are deterministic, but they demonstrate how a change in the definition of the momentum can change the couplings between the torsion and $L^{\gamma\delta}$, $S^{\gamma\delta}$ or $\rho^{\alpha}{}_{\nu} N^{\gamma\delta\nu}$. Perhaps some other redefinition of the momentum and/or the orbital or spin angular momenta would make the propagation equations deterministic. I have not been able to find such a redefinition.

But why are we allowed to redefine the momentum and angular momentum? Because all we know about the general relativistic definitions of these quantities is that they must have the correct special relativistic and Newtonian limits. For the momentum, all of the quantities P^{γ}, $\mathcal{P}^{\gamma}$, $\overline{\mathcal{P}}^{\gamma}$ and $\overline{\overline{\mathcal{P}}}^{\gamma}$ satisfy this criteria. Following equation (IV.3.12) I mentioned two more definitions of the momentum. The first (suggested by Madore) is

$$P'_{\delta} = \int t_{\delta}{}^{o} = \int g_{\delta\gamma} \; t^{\gamma o}, \qquad (51)$$

which differs from

$$P_{\delta} = g_{\delta\gamma} P^{\gamma} = g_{\delta\gamma} \int t^{\gamma o}, \qquad (52)$$

because $g_{\delta\gamma}$ is not constant. The second (halfway between P_{δ} and P'_{δ}) is the orthonormal momentum

$$\hat{P}_{\hat{\mu}} = \int t_{\hat{\mu}}{}^{o}. \qquad (53)$$

Similar ambiguities exist in the definitions of the orbital and spin angular momenta. I have a preference for the definitions using $t_{\hat{\mu}}{}^{\alpha}$ and $\mathcal{S}^{\hat{\mu}}{}_{\hat{\nu}}{}^{\alpha}$, ($\hat{\mu}$ and $\hat{\nu}$ orthonormal, α coordinate) but I have not had time to work out the equations.

The remaining topic to be discussed in this section is the choice of the coordinate system, x^{α}, on the neighborhood, W. This entails the choice of the spacelike foliation, $\Sigma(t)$, the choice of the timelike curve $X(t)$, and the choice of the spacial coordinates, δx^{α}, on each surface $\Sigma(t)$ centered at $X(t)$.

In the metric theory case, Madore and Beiglbøck proceed as follows: For each point, $y \in W$, and each future-directed timelike unit vector, $n^{\alpha} \in T_y W$, they define a spacelike surface

$$\Sigma(y,\ n) = \{x \in W : x \text{ is connected to } y \text{ by a geodesic orthogonal to } n\}, \tag{54}$$

and a momentum vector

$$P^{\alpha}(y,\ n) = \int_{\Sigma(y,n)} t^{\alpha\beta}\, d^3x_{\beta}. \tag{55}$$

Then for each $y \in W$, they define $u^{\alpha}(y) \in T_y W$ as the fixed point of the map

$$n^{\alpha} \longrightarrow P^{\alpha}(y,\ n)/|P^{\alpha}(y,\ n)|. \tag{56}$$

(Existence follows from Brouwer's fixed-point theorem. Uniqueness is assumed.) Using the abbreviations

$$\Sigma(y) = \Sigma(y,\ u(y)), \tag{57}$$

$$P^{\alpha}(y) = P^{\alpha}(y,\ u(y)), \tag{58}$$

the definition of $u^{\alpha}(y)$ says that $P^{\alpha}(y)$ is parallel to $u^{\alpha}(y)$ and orthogonal to $\Sigma(y)$. Next on each surface, $\Sigma(y)$, they introduce normal coordinates,

δx^α, centered at y, and define the angular momentum about y as

$$L^{\alpha\beta}(y) = 2 \int_{\Sigma(y)} \delta x^{[\alpha} \, t^{\beta]\gamma} \, d^3x_\gamma . \tag{59}$$

The center of mass curve (center of motion curve according to Beiglbock) is then defined as

$$X = \{y \in W : L^{\alpha\beta}(y) \, P_\beta(y) = 0\}. \tag{60}$$

That X is a unique C^1-timelike curve in *int* W follows from Brouwer's fixed-point theorem and certain assumptions about $t^{\alpha\beta}$. Let t be a future-directed affine parameter on X and abbreviate

$$\Sigma(t) = \Sigma(X(t)), \tag{61}$$

$$u^\alpha(t) = u^\alpha(X(t)), \tag{62}$$

$$P^\alpha(t) = P^\alpha(X(t)), \tag{63}$$

$$L^{\alpha\beta}(t) = L^{\alpha\beta}(X(t)). \tag{64}$$

Finally, they choose x^α as a generalized Fermi coordinate system based on the curve X(t) and the timelike vector field $u^\alpha(t)$. In that case, $P^\alpha(t)$ and $L^{\alpha\beta}(t)$ coincide with the momentum and orbital angular momentum used in the rest of this chapter. However, because of the special choice of the curve, X(t), they also satisfy

$$P^\alpha(t) = m(t) \, u^\alpha(t), \tag{65}$$

$$L^{\alpha\beta}(t) \, P_\beta(t) = 0. \tag{66}$$

In the x^α coordinate basis, equations (65) and (66) say

$$P_\alpha(t) = m(t)\, u_{\cdot\alpha} = s\ m(t)N(t)\delta^o_\alpha, \tag{67}$$

$$L^{\alpha o}(t) = 0, \tag{68}$$

which show that $\mathcal{P}^\alpha$ coincides with P^α in Corollary IV.4 and simplifies in Corollary IV.7.

It is obvious that the same construction also works in the metric-Cartan connection theory case. However, it is in no sense unique. Alternatively, one could define $\Sigma(y, n)$ using Cartan geodesics instead of Christoffel geodesics; or one could use the Cartan connection instead of the Christoffel connection to define the normal coordinates, δx^α, and the generalized Fermi coordinates, x^α. In that case the curve X(t), may change but equations (65) and (66) remain valid. Which coordinate system is better, I do not know. Perhaps another choice of coordinates could be found which would eliminate the indeterminacies in the propagation equations of Theorem IV.2. In any case, the results in Corollaries IV.1 through IV.7 and Theorem IV.2 are true for any choice of coordinates.

5. Experiments

a. Stanford-Schiff Gyroscope Experiment

My original purpose for investigating the equations of motion of a rotating body in metric-connection theories was to see whether the Schiff gyroscope experiment, presently being designed at Stanford, could detect the presence of a torsion field. The answer is no. This follows because the gyroscope has no net elementary particle spin. Hence, Corollaries IV.3 and IV.4 show that the gyroscope will not feel the torsion and will move exactly as it would in a metric theory.

b. Magnetized Gyroscope

In short, a gyroscope will feel the torsion only if its total elementary particle spin is non-zero; i.e. it is magnetized. The appropriate propagation equations would be an electromagnetic generalization of equations (IV.3.16) and (IV.3.17) of Theorem IV.2. However, to get a rough estimate of the importance of the torsion, I will ignore the electromagnetic interactions. Also, to eliminate the indeterminacies in equations (IV.3.16) and (IV.3.17) I will assume that

$$\rho^{\alpha}{}_{\nu}\, N^{\gamma\delta\nu} = 0, \tag{1}$$

and that the spin and orbital angular momenta are proportional with a constant ratio. Thus I define

$$J^{\gamma\delta} = L^{\gamma\delta} + S^{\gamma\delta}, \tag{2}$$

and assume

$$S^{\gamma\delta} = \sigma\, J^{\gamma\delta}, \qquad L^{\gamma\delta} = (1-\sigma) J^{\gamma\delta}. \tag{3}$$

After some manipulation, equations (IV.3.16) and (IV.3.17) become

$$\nabla_v p^\gamma = \frac{1}{2} J^{\beta\alpha}[(1-\sigma)\tilde{R}_{\alpha\beta}{}^{\gamma}{}_{\delta} + \sigma\hat{R}_{\alpha\beta}{}^{\gamma}{}_{\delta} - 2\sigma(1-\sigma)\lambda_{\alpha\varepsilon}{}^{\gamma}\lambda^{\varepsilon}{}_{\beta\delta}]v^\delta$$

$$+ \sigma p^\beta \lambda_{\beta\delta}{}^{\gamma} v^\delta, \qquad (4)$$

$$\nabla_v J^{\alpha\beta} = -2\, v^{[\alpha} p^{\beta]} + 2\sigma\, J^{\gamma[\alpha}\lambda^{\beta]}{}_{\gamma\delta}\, v^\delta. \qquad (5)$$

To obtain an estimate for σ, I assume that the gyroscope is an iron sphere of radius, R, rotating with angular velocity, ω, and that one electron per atom is aligned. Since the density and atomic weight of iron are $\rho = 7.86\text{g/cm}^3$ and A = 56g/mole, the mass of the gyroscope is

$$M = \frac{4}{3}\pi R^3 \rho = (\frac{R}{cm})^3\ 33\ g, \qquad (6)$$

and the total number of aligned electrons is

$$N_e = N_A\ M/A = (\frac{R}{cm})^3\ 3.7 \times 10^{23}\ , \qquad (7)$$

where $N_A = 6.02 \times 10^{23}$/mole is Avogadro's number. Hence the total spin is

$$S = \frac{1}{2}\hbar N_e = (\frac{R}{cm})^3\ 1.9 \times 10^{-4}\ \frac{g\ cm^2}{sec}\ , \qquad (8)$$

while the total orbital angular momentum is

$$L = \frac{2}{5} M R^2 \omega = (\frac{R}{cm})^5\ (\frac{\omega}{sec^{-1}})\ 13\ \frac{g\ cm^2}{sec}\ . \qquad (9)$$

Therefore,

$$\frac{\sigma}{1-\sigma} = \frac{S}{L} = (\frac{cm}{R})^2\ (\frac{sec^{-1}}{\omega})\ 1.5 \times 10^{-5}\ , \qquad (10)$$

$$\sigma = (\frac{cm}{R})^2\ (\frac{sec^{-1}}{\omega})\ 1.5 \times 10^{-5}\ . \qquad (11)$$

Returning to equation (5), we see that the torsion induced torque, $2\,\sigma\, J^{\gamma[\alpha}\,\lambda^{\beta]}{}_{\gamma\delta}\, v^{\delta}$, will only be as large as the Christoffel corrections, $2\, J^{\gamma[\alpha}\{^{\beta]}{}_{\gamma\delta}\}v^{\delta}$, if the defect, $\lambda^{\beta}{}_{\gamma\delta}$, is $1/\sigma \sim 10^5$ times as large as the Christoffel symbols, $\{^{\beta}{}_{\gamma\delta}\}$. This seems unlikely. A similar examination shows that in equation (4) the torsion induced forces are again smaller by a factor of σ than the Christoffel interactions. Thus it seems unlikely that even a magnetized gyroscope could detect a torsion field.

c. Neutron Star

A second system with both spin and orbital angular momentum is a neutron star. The following calculation again shows that the spin angular momentum is much smaller than the orbital angular momentum.

Let M_*, $M_\odot$ and m denote the masses of the neutron star, the sun and a neutron. Then the total number of neutrons is

$$N = \frac{M_*}{m} = \left(\frac{M_*}{M_\odot}\right) 1.2 \times 10^{57}. \tag{12}$$

Assuming the best case that all of the neutron spins are aligned, the total spin is

$$S = \frac{1}{2}\,\hbar\, N = \left(\frac{M_*}{M_\odot}\right)\; 6 \times 10^{29}\; \frac{\text{g cm}^2}{\text{sec.}} \;. \tag{13}$$

On the otherhand, the total orbital angular momentum is

$$L = \frac{2}{5}\, M_*\, R^2\, \omega = \left(\frac{M_*}{M_\odot}\right)\left(\frac{R}{10\text{km}}\right)\left(\frac{\omega}{\text{sec}^{-1}}\right) 8 \times 10^{44}\; \frac{\text{g cm}^2}{\text{sec.}}\;. \tag{14}$$

Therefore,

$$\frac{\sigma}{1-\sigma} = \frac{S}{L} = \left(\frac{10\text{km}}{R}\right)\left(\frac{\text{sec}^{-1}}{\omega}\right) 7.5 \times 10^{-16}\;, \tag{15}$$

which is even worse than the corresponding quantity for a magnetized gyroscope.

d. Magnet

Examining equations (8), (9), and (10) shows that the best way to increase σ is to set $\omega = 0$; i.e. to stop the rotation. But then the magnetized gyroscope is no longer a gyroscope but only a magnet. The appropriate propagation equations are (IV.4.23) and (IV.4.24) of Corollary IV.5. These are to be compared with equations (IV.4.12) and (IV.4.13) of Corollary IV.3, which are appropriate to the Stanford gyroscope.

The Stanford gyroscope is a quartz sphere of density $\rho = 2.2$ g/cm^3, radius $R = 2.0$ cm., angular velocity $\omega = 400\pi$ rad/sec and hence orbital angular moemntum $L = 1.5 \times 10^5$ g cm^2/sec. To have an equal spin angular momentum in a spherical, non-rotating, iron magnet the radius would have to be $R = 920$ cm. Even then the torsion would have to be comparable to the Christoffel symbols in order to be detectable.

e. Polarized Particle Beams and He^3 Superfluid

Two other systems with spin angular momentum but no orbital angular momentum are (i) a beam of polarized elementary particles (protons, electrons, neutrons, etc.) and (ii) a He^3 superfluid. I have not yet analyzed the propagation of momentum and spin polarization in such quantum mechanical systems.